0. [0 pt] What is your name?

1. [8 pt] Continuum Hypothesis

(a) Explain why the continuum hypothesis is important in the study of fluid dynamics

   The continuum hypothesis is important in fluid mechanics because it allows us to examine the dynamics and kinematics of a fluid at the macroscopic level without considering the molecular nature of matter.

(b) For a gas, when is the continuum hypothesis valid [Hint: think length scales]?  

   The continuum hypothesis is valid in a gas when the mean free path of the gas is much larger than the smallest length scale of the fluid flow phenomena of interest.

2. [9 pt] Write the following vector quantities in index notation.

(a) \( \vec{u} \times (\nabla \times \vec{u}) \)

\[
\vec{u} \times (\nabla \times \vec{u}) = \frac{\partial u_k}{\partial x_l} \epsilon_{klm} \epsilon_{imn}
\]

(b) \( \nabla \cdot (\nabla \times \vec{u}) \)

\[
\nabla \cdot (\nabla \times \vec{u}) = \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_k} \epsilon_{jki}
\]

(c) \( \nabla \times \nabla \phi \)

\[
\nabla \times \nabla \phi = \frac{\partial}{\partial x_i} \frac{\partial \phi}{\partial x_j} \epsilon_{ijk}
\]
3. Streamlines in a porous channel are shown. The top and bottom walls are heated to a temperature $T_1$ and $T_2$, respectively, where $T_2 > T_1$. The flow is steady. Consider the following equation:

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j}$$

a. How is the equation useful in the given problem? The equation allows us to convert between the Lagrangian and Eulerian viewpoint.

b. What is term I called?
   Material derivative of temperature

c. Describe in words what happens to the time rate of change of the temperature of the fluid particle indicated by the black circle. Which term in the equation represents this?
   Because the flow is steady, streamlines and pathlines coincide. As the particle moves along the streamline, its temperature decreases. This is represented by $\frac{DT}{Dt}$.

d. Describe in words the time rate of change of the temperature at the fixed point indicated by the open circle. Which term in the equation represents this?
   Since the flow is steady, the temperature at the point remains constant. This is represented by $\frac{\partial T}{\partial t}$.

e. What is term III called? What is the sign of term III at the fixed point indicated by the open circle? [Support your answer]
   Term III is the advection of temperature. This term can be represented by: $u_j \frac{\partial T}{\partial x_j} = u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}$. $\frac{\partial T}{\partial x} = 0$, $v > 0$ (flow is upward), and $\frac{\partial T}{\partial y} < 0$ since $T_2 > T_1$ (flow temp decreases up) therefore, $u_j \frac{\partial T}{\partial x_j} < 0$. 
4. [10 pt] A student claims that a two-dimensional flow with a non-constant density field in the form
\[ \rho = a_0 + a_1 x + a_2 x^2, \]
where \( a_0, a_1, \) and \( a_2 \) are constants, is incompressible.

a. Write the mathematical definition of incompressible flow.
\[
\frac{D\rho}{Dt} = 0
\]

b. Based on your knowledge of fluid dynamics, is the student’s claim above correct or incorrect? [Show the work to support your answer]

For the flow to be incompressible, \( \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0 \). The density is steady (no time dependence), therefore our criteria reduces to:
\[
u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} = 0
\]
The density is only a function of \( x \) (\( \rho = \rho(x) \)) \( \rightarrow \frac{\partial \rho}{\partial y} = 0 \) and our criteria is just:
\[
u \frac{\partial \rho}{\partial x} = 0
\]
and \( \frac{\partial \rho}{\partial x} = a_1 + 2a_2 x \)
so that \( \nu(a_1 + 2a_2 x) = 0 \)
This only has a solution if \( a_1 = x = 0 \) or \( a_1 = a_2 = 0 \) or \( \nu = 0 \). Therefore, this flow is not incompressible except for the trivial solutions where \( \nu = 0 \) or \( \rho = a_0 \) (i.e. a constant).
5. [10 pt] The image below illustrates the streamline pattern in a flow at a given time $t$. A fluid particle is indicated by the black circle.

a. How are the streamlines related to the stream function?

The stream function is constant along any streamline so that different streamlines can be identified by their stream function. Mathematically $d\psi = 0$ along any streamline.

b. From this image alone, can you predict the pathline of the particle indicated by the black circle?

From the figure alone it is not possible to predict the path line traced out by any particle. This is because we don’t know if it is steady or unsteady. If the flow is steady, we could predict the path line since it would coincide with the streamlines.

c. What can you say about the magnitude of the velocity at point 1 compared to that at point 2? While it is difficult to say anything definitive about the velocity from the given figure, we can deduce that the flow is accelerating in the vertical direction at 2 based on the compression of the streamlines and decelerating in the horizontal direction as it moves into the roof top boundary layer.
6. [28 pt] Streamlines of the flow in a corner is shown below. The horizontal and vertical velocity components associated with this flow are, respectively: \( u = x^2 - y^2 \) and \( v = -2xy \).

(a) Determine the components in the strain-rate tensor, \( e_{ij} \).

The strain rate tensor is given by: 
\[
e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

The only surviving terms in the strain rate tensor are:

\[
e_{11} = \frac{\partial u}{\partial x} = 2x
\]
\[
e_{22} = \frac{\partial v}{\partial y} = -2x
\]
\[
e_{12} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2} (-2y - 2y) = -2y
\]

(b) Determine the components in the vorticity vector, \( \omega_i \).

\[
\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j}
\]

The only possible vorticity component (based on above) is:

\[
\omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -2y - (-2y) = 0
\]

(c) On the image provided, draw the pathline of the fluid particle that originates at point \((x = 3, y = 2)\) at time \( t = 0 \).

The flow is steady, therefore, the path line coincides with the streamline passing through the point.
(d) Is the flow incompressible [show work]?

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \text{ for incompressible flow}
\]

\[2x + (-2x) = 0 \rightarrow 0 = 0 \rightarrow \text{yes the flow is incompressible}\]

(e) Consider an initially square particle with centroid located at \((x = 3, y = 2)\), as shown in the streamline plot above. How will the fluid particle look after a infinitesmial time \(\Delta t\) later? [Support your reasoning with calculations]

(f) Derive expressions for the components of the acceleration of a generic fluid particle.

\[a_x = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = (x^2-y^2)(2x)+(-2xy)(-2y) = 2x^3-2xy^2+4xy^2 \rightarrow a_x = 2x^3+2xy^2\]

\[a_y = \frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = (x^2-y^2)(-2y)+(-2xy)(-2x) = -2x^2y+2y^3+4x^2y \rightarrow a_y = 2y^3+2x^2y\]

(g) Explain how the momentum of the fluid particle at \((x = 3, y = 2)\) is changing with respect to time. [Support your answer with calculations]

The velocity at the desired point can be calculated directly as: \(u = (3)^2 - (2)^2 = 5\) and \(v = -2(3)(2) = -12\).

The acceleration can be calculated from part f) as \(a_x = 2(3)^3 + 2(3)(2)^2 = 78\) and \(a_y = 2(2)^3 + 2(3)^2(2) = 52\).

From this we can deduce that the particle is moving toward the right and downward along the streamline and accelerating in the horizontal and decelerating in the vertical.
7. [23 pt] Consider planar flow in a channel with porous walls. A constant vertical velocity $V_w$ exists at the top and bottom walls as shown. Flow is driven through the channel by a pressure gradient $\partial P/\partial x = -K$.

![Image of flow in a channel](image)

a. Use the continuity equation to find the vertical velocity $v$. [State assumptions]

Assumptions: 1) fully-developed, 2) incompressible

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

fully developed therefore $\partial / \partial x = 0$ so that $\rightarrow v = \text{const}$. Applying our boundary condition that $v(y = 0) = V_w \rightarrow \text{const} = V_w \rightarrow v = V_w$.

b. Simplify the Navier-Stokes equation (x-component ONLY) for this flow and provide the appropriate boundary conditions. [State assumptions]

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \mu \frac{\partial^2 u}{\partial y^2}$$

assuming steady (SS), fully developed (FD), incompressible flow:

$$\rho \frac{\partial u}{\partial t}^{\text{SS}} + \rho u \frac{\partial u}{\partial x}^{\text{FD}} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2}^{\text{FD}} + \mu \frac{\partial^2 u}{\partial y^2}$$

using $v = V_w$ and $-\partial P/\partial x = K$ we have:

$$\rho V_w = K + \mu \frac{\partial^2 u}{\partial y^2}$$

With boundary conditions: $u(y = 0) = 0$ and $u(y = h) = 0$
c. For the case where \( V_w = 0 \), explain the role of viscous diffusion in the fluid dynamics. 

In this case, the channel acts like a basic "solid wall" channel. The walls act as momentum sinks. The impact of this sink diffuses vertically (away from the walls) towards the centerline due to molecular interactions between neighboring fluid particles. Fluid near the wall slows down and fluid at the channel centerline speeds up to satisfy conservation of mass as a result of this process.

d. How would you expect the streamline pattern to change if the viscosity of the fluid was increased? [Suggestion: write the acceleration term in (b) from the Lagrangian viewpoint, then consider the motion of a fluid particle.]

\[
\rho \frac{Du}{Dt} = K + \mu \frac{\partial^2 u}{\partial y^2}
\]

If the viscosity increases, then the rate of change of stream wise momentum (LHS in above equation) will increase. The vertical velocity \( (V_w) \) remains constant and therefore, the streamlines will appear to flatten out. If \( \mu \) is large enough the streamlines will flatten out and become horizontal.

e. [Extra Credit] Solve for the horizontal velocity component.

\[
\mu \frac{\partial^2 u}{\partial y^2} - \rho V_w \frac{\partial u}{\partial y} = -K \quad \text{with} \quad u(y = 0) = u(y = h) = 0
\]

Homogeneous part:

\[
\mu \frac{\partial^2 u}{\partial y^2} - \rho V_w \frac{\partial u}{\partial y} = 0
\]

characteristic polynomial: \( \mu \lambda^2 - \rho V_w \lambda = 0 \rightarrow \lambda_1 = 0, \lambda_2 = \frac{\rho V_w}{\mu} \)

homogeneous solution: \( u_h = c_1 + c_2 \exp \left( \frac{V_w y}{\nu} \right) \)

particular solution: \( u_p = c_3 y + c_4 \) (guess this) plug it into ODE \( \rightarrow c_4 = 0, \ c_3 = \frac{K}{\rho V_w} \)

The total solution is: \( u = u_h + u_p = c_1 + c_2 \exp \left( \frac{V_w y}{\nu} \right) + \left( \frac{V_w y}{\nu} \right) y \)

From the BCs: \( -c_1 = c_2 = \frac{K h}{\rho V_w} \left( \frac{1}{1 - \exp(V_w y / \nu)} \right) \rightarrow \)

\[
\frac{K h}{\rho V_w} \left[ \frac{y}{h} - \frac{1 - \exp(V_w y / \nu)}{1 - \exp(V_w h / \nu)} \right]
\]