Derivation of equation for $u_i' u_i'$

\[
\frac{u_i' \partial u_i}{\partial t} = \ldots .
\]

\[\frac{u_i' \partial u_i}{\partial t} = \ldots .
\]

add together

\[
\frac{u_i' \partial (u_i' + u_i')}{\partial t} + \frac{u_i' \partial (u_i' + u_i')}{\partial t} = \ldots .
\]

\[
\frac{u_i' \partial u_i'}{\partial t} + \frac{u_i' \partial u_i'}{\partial t} = \ldots .
\]

\[
\frac{\partial u_i}{\partial t} = \ldots .
\]

Let $u_j' u_j' = R_{ij}$ The final full equation is:

\[
\frac{\partial R_{ij}}{\partial t} + u_k \frac{\partial}{\partial x_k} R_{ij} = P_{ij} + T_{ij} - D_{ij} - \frac{2}{\partial x_k} J_{ijk}
\]

\[P_{ij} = - ( R_{ik} \frac{\partial u_i}{\partial x_k} + R_{ik} \frac{\partial u_i}{\partial x_k} ) \quad \text{Production, closed}
\]

\[J_{ijk} = - \nu \frac{\partial}{\partial x_k} R_{ij} + u_i' u_j' u_k' + \frac{1}{\rho} (u_i' p' \delta_{ik} + u_i' p' \delta_{ik})
\]

\[D_{ij} = \frac{\partial}{\partial x_k} \frac{\partial u_i}{\partial x_k} + \frac{\partial}{\partial x_k} \frac{\partial u_j}{\partial x_k}
\]

\[T_{ij} = \frac{1}{2} p' \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]
\( \mathbf{J}_{ijk} - \) Transport, requires modeling

\[ \frac{2}{\varepsilon_{x_k}} u_i' u_j' u_k' \leq \text{turbulent transport of } u_i' u_j' \]

\[ u_i' p' S_{i,k} + u_i' p' \delta_{i,k} \quad \text{-- Not really know how this part behaves} \]

DNS suggests it is small. So either ignore or absorb into \( u_i' u_j' u_k' \)

Modeling of \( u_i' u_j' u_k' \)

Simplest \( u_i' u_j' u_k' \sim \frac{\partial u_i' u_j'}{\partial x_k} \leq \text{gradient transport} \)

but \( u_i' u_j' u_k' \) is "rotationally invariant" (symmetric in all indices)

So take form:

\[ C_{ijk} = \frac{2}{3} C \left( \frac{\partial u_i' u_j'}{\partial x_k} + \frac{\partial u_i' u_k'}{\partial x_j} + \frac{\partial u_j' u_k'}{\partial x_i} \right) \]

\[ \uparrow \]

\( v_t \)

(Not only model -- variations exist)
Dissipation:

\[ D_{ij} = 2 \nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \]

Assuming \( D_{ij} \) is a small scale process, \( D_{ij} \) should be isotropic.

\[ D_{ij} = \frac{\nu}{3} \delta_{ij} \]

\[ \varepsilon = \nu \frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k} \] obtained from \( \varepsilon \) equation.

Not isotropic near walls - add damping & non-isotropy.

Pressure Strain

The pressure & velocity correlations that appear on the Reynolds stress equation can be decomposed in several ways. As done here (which is common), we are left with a "pressure-strain term":

\[ \frac{1}{2} p' \left( \frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i} \right) = T_{ij} \]

This must be modeled. What to do?

To gain some understanding, an equation for the fluctuating pressure can be derived.

Before doing this, note \( T_{ii} = 0 \) (for incompressible flow) so it does not contribute to kinetic energy. It redistributes energy between various Reynolds stress components (See Pope, pg 353).
To derive an equation for $p'$, note the following:

Take full N.S. equation:

$$\frac{Du_i'}{Dt} + u_j' \frac{\partial u_i'}{\partial x_j} + u_i' \frac{\partial u_j'}{\partial x_j} - \frac{\partial u_ju_i'}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_i \partial x_j}$$

Subtract mean momentum:

For an incompressible flow, this gives

$$\frac{Du_i'}{Dt} + u_j' \frac{\partial u_i'}{\partial x_j} + u_i' \frac{\partial u_j'}{\partial x_j} - \frac{\partial u_ju_i'}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p'}{\partial x_i} + \nu \frac{\partial^2 u_i'}{\partial x_i \partial x_j}$$

Next, take $\nabla \cdot$:

This gives

$$\nabla \cdot u' = -\frac{\partial (u_ju_i' - u_iu_j')}{\partial x_i} - 2 \frac{\partial u_i'}{\partial x_i} \frac{\partial u_i'}{\partial x_i}$$

This can be solved by integrating along a line and applying Green's theorem (See pg 19 in Pope).

Solution is:

$$\frac{\mathcal{E}^k}{S} = \frac{1}{4\pi} \int_{Vol.} \left\{ \frac{\partial^2 (u_j u_i' - u_i u_j')}{\partial x_i \partial x_j} + 2 \frac{\partial u_i'}{\partial x_i} \frac{\partial u_i'}{\partial x_i} \right\} \frac{dV}{p}$$

To get the pressure-strain correlation, multiply the above by $\frac{\partial u_i'}{\partial x_i} \frac{\partial u_j'}{\partial x_j}$ and average

$$\frac{\partial}{\partial x_i} \left( \frac{\partial u_k'}{\partial x_j} + \frac{\partial u_j'}{\partial x_k} \right) = \frac{1}{4\pi} \int_{Vol.} \left[ \left( \frac{\partial^2 u_k u_i'}{\partial x_k \partial x_i} \right) - \left( \frac{\partial u_i'}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} \right) \right] \frac{dV}{p}$$

$$+ \frac{1}{4\pi} \int_{Vol.} \left[ 2 \frac{\partial u_k}{\partial x_m} \frac{\partial u_m'}{\partial x_k} \right] \left( \frac{\partial u_i'}{\partial x_i} + \frac{\partial u_i'}{\partial x_i} \right) \frac{dV}{p}$$
The standard interpretation of this is that there is a "rapid" and "slow" part.

The first term contains only turbulence quantities and is termed "return to isotropy" or "slow distortion" (Ti$_i^S$).

For $i=j$, $T_{ii}^S = 0 = T_{ij}$.

Things we say in modeling 2nd order closure:
- Turbulence quantities are local functions of $u'_i u'_j, k, e, u_i$, etc.
- Consistent in symmetry
- Turbulent phenomena characterized by single scale based on $k, e$
- Small eddies isotropic

So say $T_{ij}^S = -\frac{C}{t} u'_i u'_j$ and decay

but require $T_{ii}^S = 0$

so $T_{ij}^S = -\frac{C}{t} \left( u'_i u'_j - \frac{2}{3} \delta_{ij} k \right)$

$$=-\frac{C}{k} \left( u'_i u'_j - \frac{2}{3} \delta_{ij} k \right) \quad \text{(Pope eq. 11.24)}$$

This is earliest model (Rotta, 1951)

Advanced ideas: Real process of redistribution is nonlinear & modeling should reflect this.

Pope 11.3.3


C has taken on wide range of values.
"Rapid" Pressure strain  "rapid distortion" "rapid return to isotropy"

\[ T_{ij}^r = \frac{1}{4\pi} \int \left[ 2 \left( \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} \right) \left( \frac{\partial u_j}{\partial x_l} + \frac{\partial u_i}{\partial x_l} \right) \right] \frac{\mathrm{d}V}{r^2} \]

Named "rapid" since there is an instantaneous response in this correlation to mean velocity gradients.

Simplest modeling approach

Approximate \( T_{ij}^r \) by shrinking down size of integration volume:

\[ T_{ij}^r = C \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_l} \left( \frac{\partial u_i}{\partial x_l} + \frac{\partial u_j}{\partial x_l} \right) \frac{\ell^3}{l} \]

The assumption here is all turbulent transport quantities are local functions of \( k, \varepsilon, u_i u_i', \bar{u}, \) etc.

For proper symmetry, \( T_{ii}^r = 0 \), and correct scaling we can write:

\[ T_{ij}^r = C \left( \frac{\partial u_i}{\partial x_m} \frac{u_i u_i'}{x_m} + \frac{\partial u_i}{\partial x_m} \frac{u_j u_j'}{x_m} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_m} \frac{u_k u_k'}{x_m} \right) \]

\[ -P_{ij} \]

\[ T_{ij}^r = -C \left( P_{ij} - \frac{2}{3} \delta_{ij} P \right) \]

For high strain

\[ (2 \delta_{ij} \delta_{ij})^{\frac{1}{2}} >> \frac{\varepsilon}{k} \quad \Rightarrow \quad T_{ij}^r \quad \text{dominates} \]

See RDT (Pope, pg. 404)

General modeling more sophisticated than above.
Boundary Conditions:

At walls:

No slip applies

Problems: 1) Steep gradients \(\Rightarrow\) require very high resolution

2) Viscous effects important \& high Re turbulence models (i.e. \(u^+\)) not applicable

A Solution: Use empirical laws to connect wall conditions (i.e. wall shear stress) to dependent variables outside viscous sublayer

From momentum equation, near wall velocity profile can be approximated by

\[
U^+ = \frac{1}{K} \ln y^+ + C
\]

\[
U^+ = \frac{U}{U^*}, \quad U^* = \left(\frac{\tau_w}{\rho_c}\right)^{1/2}, \quad \tau_w = \mu \frac{dU}{dy} \text{ wall}
\]

\[
\gamma^+ = \frac{U^* y}{V}
\]

Very near wall (\(\gamma^+ \lesssim 10\)), viscous effects dominate

For \(\gamma^+ > 10\), can write \(U^+ = \frac{1}{K} \ln E y^+\)

\(E = 9\) for smooth walls

Solve regular \(k, \epsilon\) equations in turbulent zone

Use wall \(f^+\) for near wall \& to get \(u\) at first mesh point

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Solution B:

Wall models not always adequate

e.g. separated flows
unsteady flows
transitional Re

Cannot predict things directly near wall

Low Reynolds # approach using wall damping

Typical set

\[-u_i'u_j' = \nu_t \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} k\]

Where \( \nu_t = C_{nu} \frac{f_m}{E} \)

\( E = \bar{E} + D \)

Tabulated for various models in Chen & Jow pg. 117-118

\( \bar{E} = \nu \frac{\partial}{\partial x_j} (C_n \frac{f_m}{E} \frac{\partial k}{\partial x_j} + \nu \frac{\partial k}{\partial x_j}) - u_i'u_j' \frac{\partial u_i}{\partial x_j} - \bar{E}\)

\( \frac{\partial \bar{E}}{\partial t} = \frac{2}{\partial x_j} \left( C_n \frac{f_m}{E} \frac{\partial E}{\partial x_j} + \nu \frac{\partial E}{\partial x_j} \right) - C_{e_1} f_1 \frac{\bar{E}}{k} u_i'u_j' \frac{\partial u_i}{\partial x_j} \)

\( f_m \)

\( f_1 \) involve \( y, y^+, \nu, k \)

\( f_2 \) several variants exist