LES of Turbulent Flows: Lecture 10
(ME EN 7960-008)

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LES filtered Equations for incompressible flow

•Mass: $$\frac{\partial \tilde{u}_i}{\partial x_i} = 0$$ ⓐ

•Momentum: $$\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} + F_i$$ ⓑ

•Scalar: $$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial \tilde{u}_i \tilde{\theta}}{\partial x_i} = \frac{1}{Sc\,\mathrm{Re}}\frac{\partial^2 \tilde{\theta}}{\partial x_i^2} - \frac{\partial q_i}{\partial x_i} + Q$$

•SFS stress: $$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$$

•SFS flux: $$q_j = \widetilde{u_j \theta} - \tilde{u}_j \tilde{\theta}$$

• we've talked about variance (or energy) when discussing turbulence and filtering

• when we examined application of the LES filter at scale Δ we looked at the effect of the filter on the distribution of energy with scale.

• A natural way to extend our examination of scale separation and energy is to look at the evolution of the filtered variance or kinetic energy

The filtered kinetic energy equation

- **<u>filtered kinetic energy equation</u>** for incompressible flow
    - -We can define the total filtered kinetic energy by: $\tilde{E} = \frac{1}{2}\widetilde{u_i u_i}$
    - -We can decompose this in the standard way by:

$$\tilde{E} = \underbrace{\tilde{E}_f}_{\text{Resolved Kinetic energy}} + \underbrace{k_r}_{\text{SFS Kinetic energy}}$$

- -The <u>SFS kinetic energy</u> (or residual kinetic energy) can be defined as:

$$k_r = \tfrac{1}{2}\left(\widetilde{u_i u_i} - \tilde{u}_i \tilde{u}_i\right)$$

(see Pope pg. 585 or Piomelli et al., Phys Fluids A, 1991)

- -The <u>resolved (filtered) kinetic energy</u> is then given by:

$$\tilde{E}_f = \tfrac{1}{2}\tilde{u}_i \tilde{u}_i$$

The filtered kinetic energy equation

- We can develop an equation for $\tilde{E}_f$ by multiplying equation b on page 2 by $\tilde{u}_i$:

$$\boxed{\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t}} + \boxed{\tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}} = -\boxed{\tilde{u}_i \frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i}} + \nu \tilde{u}_i \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j}$$

- Applying the product rule to the terms in the squares:

$$\frac{\partial \tilde{u}_i \tilde{u}_i}{\partial t} = \tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t} \Rightarrow \boxed{\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t} = \tfrac{1}{2} \frac{\partial \tilde{u}_i \tilde{u}_i}{\partial t}}$$

$$\tilde{u}_j \frac{\partial \tilde{u}_i \tilde{u}_i}{\partial x_j} = \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \Rightarrow \boxed{\tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \tfrac{1}{2} \tilde{u}_j \frac{\partial \tilde{u}_i \tilde{u}_i}{\partial x_j}}$$

$$\frac{\partial \tilde{P} \tilde{u}_i}{\partial x_i} = \tilde{P} \overset{0 \text{ (eqn a)}}{\cancel{\frac{\partial \tilde{u}_i}{\partial x_i}}} + \tilde{u}_i \frac{\partial \tilde{P}}{\partial x_i} \Rightarrow \boxed{\tilde{u}_i \frac{\partial \tilde{P}}{\partial x_i} = \frac{\partial \tilde{P} \tilde{u}_i}{\partial x_i}}$$

- Using our definition of $\tilde{E}_f$:

$$\frac{\partial \tilde{E}_f}{\partial t} + \tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{u}_i \tilde{P}}{\partial x_i} + \underbrace{\nu \tilde{u}_i \frac{\partial^2 \tilde{u}_i}{\partial x_j^2}}_{\triangle} - \underbrace{\tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j}}_{\triangledown}$$

The filtered kinetic energy equation

- term △ :

$$\frac{\partial^2 \tilde{u}_i \tilde{u}_i}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\tilde{u}_i \tilde{u}_i\right] \overset{\text{Product rule}}{=} \boxed{\frac{\partial}{\partial x_j}\left[2\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial x_j}\right] \overset{\text{Product rule}}{=} 2\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j} + \underbrace{2\tilde{u}_i \frac{\partial^2 \tilde{u}_i}{\partial x_j^2}}}$$

Looks just like △ (without ν)

- using squared equation and divide by 2 and multiplying by ν:

$$\nu \tilde{u}_i \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} = \underbrace{\nu \frac{\partial}{\partial x_j}\left[\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial x_j}\right]} - \underbrace{\nu \frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j}} \qquad \text{recall that } \tilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)$$

Uses symmetry of $\tilde{S}_{ij}$ and tensor contraction →

$$2\nu \frac{\partial}{\partial x_j}\left[\tilde{u}_i \tilde{S}_{ij}\right] - \underbrace{2\nu \tilde{S}_{ij}\tilde{S}_{ij}}_{\varepsilon_f}$$

- term ▽ :

$$\frac{\partial \tilde{u}_i \tau_{ij}}{\partial x_j} = \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j} + \tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j} \Rightarrow \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j} = \frac{\partial \tilde{u}_i \tau_{ij}}{\partial x_j} - \underbrace{\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j}}_{\tau_{ij}\tilde{S}_{ij} = \Pi}$$

- Combining everything back together:

$$\frac{\partial \tilde{E}_f}{\partial t} + \tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \tilde{u}_i \tilde{p}}{\partial x_j} - \frac{\partial \tilde{u}_i \tau_{ij}}{\partial x_j} - 2\nu \frac{\partial \tilde{u}_i \tilde{S}_{ij}}{\partial x_j} - \varepsilon_f - \Pi$$

"storage" of $\tilde{E}_f$

advection of $\tilde{E}_f$

pressure transport

transport of SFS stress τ_{ij}

transport of viscous stress

dissipation by viscous stress

SFS dissipation

Transfer of energy between resolved and SFSs

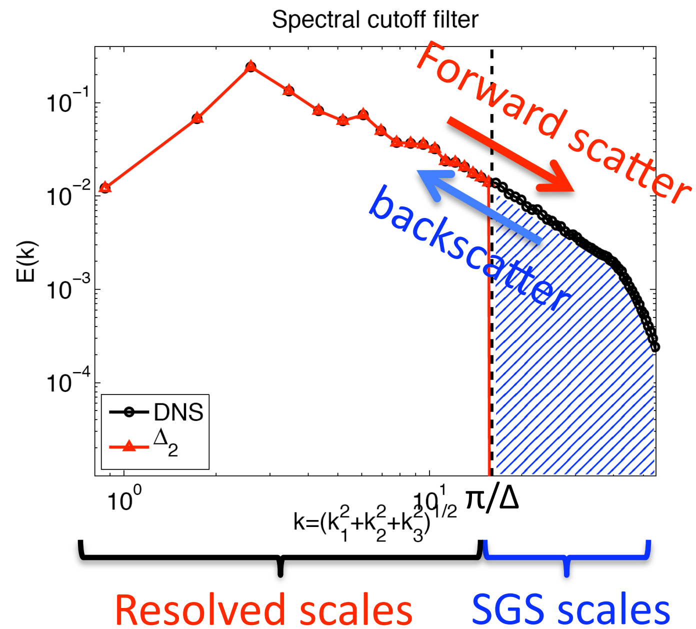
- The **SFS dissipation** Π in the resolved kinetic energy equation is a sink of resolved kinetic energy (it is a source in the k_r equation) and represents the transfer of energy from resolved SFSs. It is equal to:

$$\boxed{\Pi = -\tau_{ij}\tilde{S}_{ij}}$$

- It is referred to as the SFS dissipation as an analogy to viscous dissipation (and in the inertial subrange Π = viscous dissipation).
- On average Π **drains energy** (transfers energy down to smaller scale) from the resolved scales.
- Instantaneously (locally) Π can be positive **or** negative.

  -When Π is negative (transfer from SFS➔Resolved scales) it is typically termed **backscatter**

  -When Π is positive it is sometimes referred to as **forward scatter**.



Transfer of energy between resolved and SFSs

- Its informative to compare our resolved kinetic energy equation to the mean kinetic energy equation (derived in a similar manner, see Pope pg. 124; Stull 1988 ch. 5)

$$\frac{\partial\langle E\rangle}{\partial t}+\langle u_i\rangle\frac{\partial\langle E\rangle}{\partial x_j}+\frac{1}{\rho}\frac{\partial\langle u_i\rangle\langle P\rangle}{\partial x_j}-\frac{\partial}{\partial x_j}2\nu\langle u_i\rangle\langle S_{ij}\rangle=-P-\langle\varepsilon\rangle$$

shear production $=\langle u_i'u_j'\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}$

mean dissipation $=2\nu\langle S_{ij}\rangle\langle S_{ij}\rangle$

- For high-Re flow, with our filter in the inertial subrange:

$$\langle\tilde{E}_f\rangle=\langle E\rangle$$

- The dominant sink for $\langle\tilde{E}_f\rangle$ is Π while for $\langle E\rangle$ it is $\langle\varepsilon\rangle$ (rate of dissipation of energy). For high-Re flow we therefore have:

$$\langle\Pi\rangle\approx\langle\varepsilon\rangle$$

- Recall from K41, $\langle\varepsilon\rangle$ is proportional to the transfer of energy in the inertial subrange

➔Π will have a strong impact on energy transfer and the shape of the energy spectrum in LES.

- Calculating the correct average Π is another necessary (but not sufficient) condition for an LES SFS model (to go with our N-S invariance properties from Lecture 7).