9 TURBULENCE MODELING

The statistical treatment for turbulent fluid flow is based on a decomposition of the random fluctuating variables (velocity, pressure, temperature, etc.) into a mean component, and a fluctuating component, as first suggested by O. Reynolds. From an engineering point of view, this idea is entirely satisfactory as we are rarely interested in the details of the turbulence. Usually, it is only a knowledge of the mean values (mean transport rates) that is wanted for most design purposes. The turbulence, however, can significantly alter the mean behavior of the flow (as seen in the previous section), and these effects must be accounted for in any description of turbulent flow.

It has been previously demonstrated that the exact solution to the equations of motion is not possible for most turbulent flows of interest. This is true for both analytic and numerical approaches. In order to provide tools for engineering use, it is generally necessary to resort to some type of approximate analysis in which the effects of turbulence are not treated exactly, but are handled by means of turbulence *models*. By resorting to treating turbulence by means of models, we are sacrificing an ability to study the dynamics of turbulence from the governing equations. Instead, we are assuming something about the behavior of the turbulence, and incorporating this expected behavior into our models. It is of course the hope that the effects of turbulence are faithfully incorporated into these models.

The earliest turbulence models were simply correlations of experimental results. These types of methods are not very useful as they do not contain much physics and can only be used to predict the flow that generated the experimental results. More general methods have therefore been adopted which are based on the conservation equations. Owing to the hopelessness of obtaining exact solutions to these equations, statistical approaches have been developed for their solution. Unfortunately, our lack of understanding of turbulence still requires that all models presently used in practical engineering applications reduce to correlations with experimental data at some level. However, by building as much physics as possible into the models it is hoped that the models will have applicability to a wider range of problems and that the exact functional relationships that need to be specified by experimental data will be more universal.

9.1 The Closure Problem

The equations for the mean flow (Eq. 8.9) derived previously are exact as no approximations have been made. (The equations were derived subject to our definitions regarding decomposing a variable into mean and fluctuating components and the definition of our averaging process.) We have used these equations to provide information on how the turbulence affects the mean flow and to identify some of the mechanisms of turbulent transport. However, a fundamental problem associated with the statistical treatment of turbulence is revealed in the equations for the mean properties derived in **§8**. Through the process of decomposing the flow into its mean and fluctuating components, and averaging the governing equations, we have ended up with equations that contain more unknowns than before the averaging procedure was applied. No additional equations have been added so the equations no longer constitute a closed set. This "closure" problem is the central stumbling block in most all predictive methods for turbulent flow. As seen in Eq. 8.10, these additional unknowns (the Reynolds stresses) take the form of correlations between the fluctuating quantities. In order to close the set of equations given by Eq. 8.10, it is necessary to introduce empirical models to describe the behavior and effects of the Reynolds stresses.

One potential solution to the problem is to derive a set of transport equations that describe the time evolution of the Reynolds Stresses. This can be done (we will derive these equations later in this section), but higher order moments appear in the equations for the Reynolds Stresses. A hierarchy of equations can be derived for the unclosed terms, but closure is never achieved. Equations for the second order moments contain third order moments; equations for the third order moments contain forth order moments, and so on. At some level in this hierarchy, approximations must be introduced to close the equations. Providing a realistic closure has been a dominant motivation in turbulence research.

Hundreds of turbulence models have been suggested in the literature over the past 50 years. We will obviously not try to discuss each of these. Rather, we will attempt to clarify the physical insight that has (or has not) gone into the development of the models. Picking the best turbulence model for a particular application is not a simple matter. The complexity of the model and the amount of computer time required will vary from application to application. Here we hope to provide a flavor of the different turbulence models and point out their strengths and weaknesses.

For those interested in reviewing a rather comprehensive discussion on a wide range of turbulence models used in CFD applications, the text of Wilcox [2] is recommended.

9.2 Reynolds Stress Modeling

Let us first consider the equation for the mean motion derived in **§8**:

$$\rho\frac{\partial \bar{u}_i}{\partial t} + \rho\frac{\partial \bar{u}_i\bar{u}_j}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\overline{\sigma_{ij}} - \overline{u_i'u_j'}\right) \tag{8.10}$$

As just discussed, the presence of the Reynolds stress makes the set of equations described by Eq. 8.10 unsolvable in their present form. Although this equation is exact, there is no way of determining the value of the Reynolds Stresses.

The earliest ideas for modeling the Reynolds stresses that appear in Eq. 8.10 still make up a big part of most turbulence models in use today. In an analogy to the viscous stresses in laminar flows, Boussinesq [3] suggested that the turbulent stresses should be represented as a function of the mean velocity gradients. The "constant" (or more appropriately, function) of proportionality has been called the *eddy viscosity*. In general, the eddy viscosity representation of the Reynolds stress is

$$-\overline{u_i'u_j'} = \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{9.27}$$

where k represents the total turbulent kinetic energy. The last term on the RHS of Eq. 9.27 is to ensure that the sum of the diagonal components of the Reynolds Stress tensor equal the turbulent kinetic energy. The components $\overline{u_1'^2}$, $\overline{u_2'^2}$, and $\overline{u_3'^2}$ act as normal stresses and thus act like a pressure force. Substituting Eq. 9.27 into the equation of motion gives:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial}{\partial x_j} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \tag{9.28}$$

Because the turbulent kinetic energy acts like a normal stress in this application, we can absorb it in the pressure term in Eq. 9.28:

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial (\bar{p} + 2/3k)}{\partial x_i} + \frac{\partial}{\partial x_i} \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) \tag{9.29}$$

Because k can be absorbed in p, it is not necessary to compute it explicitly. You will look at this more carefully in a homework problem.

At first glance, it appears that using an eddy viscosity should be a reasonable approach to modeling the turbulent stresses. One of the major characteristics of turbulence which we have discussed earlier is its enhanced mixing (or diffusive) properties. Applying an *apparent* turbulent viscosity given in Eq. 9.27 by ν_t seems to make good sense on physical grounds. Determining this turbulent, or eddy viscosity is then an interesting problem and one of crucial importance.

9.3 Modeling the Eddy Viscosity

By modeling the effects of turbulence by an eddy viscosity we are sacrificing any ability to study fundamental dynamics of turbulent flow using this model. The effects of turbulence are treated by a model that has an assumed physical behavior and whose "correct" quantitative behavior will ultimately rely on empirical correlations with experimental data. However, the goal is to build as much physics as possible into the models and thereby hopefully achieve a more general applicability.

The earliest models used constant values for the eddy viscosity. Models based on that assumption do not perform well and won't be discussed here.

The next step in complexity was achieved by further extensions of the analogy between molecular viscosity and eddy viscosity. Based on our knowledge of kinetic theory of gasses, we will derive some fairly simple expressions for the eddy viscosity (first put forth by Prandtl [4]). Then we will show why the arguments that lead to these expressions can't be precisely valid, but will point out situations where they still seem to work quite well. Finally, alternatives leading to more advanced models will be derived in following subsections. It is not our goal in this class to take the "best" available technology in turbulence modeling and apply it to problems of engineering interest, but to study a variety of developments in turbulence modeling as a learning tool to try to understand more about fluid mechanics and turbulence.

The basic idea behind early expressions for the eddy viscosity was to consider the eddies in a turbulent flow to behave similarly to molecules in a gas. A fluid exhibits viscous effects as a result of molecular collisions which result in an exchange of momentum. Let us now think of turbulent eddies as individual entities which also collide with each other and exchange momentum. The molecular viscosity in a gas is proportional to the speed of the molecules (which is a function of temperature) and the mean free path between molecular collision. Relating these same concepts to the eddy viscosity, we will assume that the eddy viscosity is going to be proportional to a characteristic velocity of the turbulence, and some length scale, termed the mixing length by Prandtl, L_m:

$$\nu_t \propto V L_m \tag{9.30}$$

Next an estimate for V and L_m are still needed. When Prandtl formulated this "mixing length" model he considered a shear layer with only one mean velocity component, $\bar{u}_1$. The only mean velocity gradient is then $\partial\bar{u}_1/\partial x_2$ and the only contribution to the Reynolds stress tensor in this case is $\overline{u_1'u_2'}$. With this simplified configuration Prandtl then suggested that the mean turbulent velocity fluctuation could be represented by:

$$V = L_m|\frac{\partial \bar{u}_1}{\partial x_2}| \tag{9.31}$$

Using Eq. 9.31, the eddy viscosity can be written as:

$$\nu_t = L_m^2|\frac{\partial \bar{u}_1}{\partial x_2}| \tag{9.32}$$

The Prandtl mixing length theory thus relates the eddy viscosity to the mean velocity gradient and the mixing length. The determination of the mixing length still remains.

The determination of the mixing length relies on experimental measurements. Because of the original assumptions of a shear layer with only one nonzero mean velocity component, the mixing length models have been most successful for thin shear layers where the mixing length can often be treated as constant across the shear layer and proportional to the width of the layer at any location. For various flow configurations, the values of the mixing length is given in Table 9.1. In this table δ is the local width or thickness of the shear layer.

For a wall bounded flow such as a boundary layer on a flat plate or the walls of a duct, the expression for the mixing length has more complexity and must be modified. This is because there are different regions in the boundary layer where the mixing length will take on different values. For example, in a typical boundary

Table 9.1: Mixing Lengths For Various Configurations

Flow Configuration	L_m
Plane Mixing Layer	0.07δ
Plane Jet	0.09δ
Round Jet	0.075δ
Plane Wake	0.07δ

layer, three different regions are usually identified: An inner layer where viscous shear dominates, an outer layer where turbulent shear dominates, and an intermediate, or overlap region where both effects can play a role. If you have had a boundary layer or viscous flow course, this identification should be familiar to you. In order to account for this, different expression for the mixing length are usually used in each region. The inner layer is usually very thin and in this region the mixing length is given by the distance from the wall, $L_m = y$. In the intermediate region, a value for L of $L_m = cy$ can be used where c is a constant that must be determined. Finally, in the outer region, where turbulence dominates, a constant value for L_m is used.

The mixing length models are not very successful for numerous types of flows encountered in practice. For complex flows, this is the case mainly because of the difficulty in determining the mixing length.

9.3.1 Conceptual Difficulties With The Eddy Viscosity - Molecular Viscosity Analogy

The eddy viscosity concept cannot be fundamentally correct. Models for the molecular viscosity based on the kinetic theory of gasses have two important assumptions: the molecules retain their shape (the collisions are elastic), and the mean free path between particle collisions is large compared with the molecular size. Neither of these statements holds true for turbulent eddies. The largest eddies in the flow are of the same dimensions as the flow geometry, so that their mean free paths cannot be large. Also, the interaction between eddies causes their structure to change. Since the molecular collisions are elastic, they do not loose energy during the collisions. The eddies in a turbulent flow, however, require a steady input of energy to maintain themselves. Never the less, the eddy viscosity models are the most popular in use and their performance has been OK. The eddies do exchange momentum with each other, and instead of treating Eq. 9.27 as a physically derived law, it should can be considered simply a *definition* of the eddy viscosity.

9.4 One Equation Models

In order to improve on the mixing length model discussed above, it is necessary include more physics into our models. More sophistication can be put into a turbulence model by including a transport equation for some of the turbulence models. We will still be working within the eddy viscosity concept.

If we are to describe the velocity fluctuations by a single scale, the most realist velocity scale is $k^{\frac{1}{2}}$ where $k = \frac{1}{2}\overline{u_i'u_i'}$ is the turbulent kinetic energy. Using $k^{\frac{1}{2}}$ as a velocity scale the following expression for the eddy viscosity is obtained:

$$\nu_t = C_\mu k^{\frac{1}{2}} L \tag{9.33}$$

C_μ is a "constant" whose value must be determined experimentally. The value of k throughout the flow can be determined by a transport equation for k. We derived such an equation earlier (Eq. 8.19). For completeness we present this equation below:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'u_i'}\right) + \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'}\right) = & - \frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j'u_j'\right)} - \overline{u_i'u_j'}\frac{\partial \bar{u}_j}{\partial x_i} \\ & + \nu\frac{\partial}{\partial x_i}\overline{u_j' 2s_{ij}} - \nu\overline{2s_{ij}\frac{\partial u_j'}{\partial x_i}}\end{aligned} \tag{8.19}$$

The physical interpretation of the various terms in this equation are given in **§8.3**. In its present form, Eq. 8.19 is not of much use to us as there are terms containing triple moments that we do not have any information on. Neither do we have a transport equation (at this point) to compute their behavior. This is just another manifestation of the closure problem. Namely, the turbulent diffusion (the first term on the RHS of 8.19) and the dissipation (the last term on the RHS of 8.19) must be modeled. So by introducing an equation to describe the transport of the turbulent kinetic energy, it is necessary to model additional terms in order to obtain closure.

The last two terms in Eq. 8.19 (the viscous terms) are often written in a different form. Expanding out s_{ij} we can write:

$$\begin{aligned} 2\overline{\frac{\partial}{\partial x_i}u_j' s_{ij}} - 2\nu\overline{s_{ij}\frac{\partial u_j'}{\partial x_i}} &= \overline{\frac{\partial}{\partial x_i}u_j'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} - \overline{\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\frac{\partial u_j'}{\partial x_i}} \\ &= \frac{\partial}{\partial x_i}\overline{u_j'\frac{\partial u_i'}{\partial x_j}} + \frac{\partial}{\partial x_i}\frac{1}{2}\frac{\partial}{\partial x_i}\overline{k^2} - \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}} - \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}} \end{aligned} \tag{9.34}$$

For an incompressible fluid we have

$$\overline{u_j\left(\frac{\partial^2 u_i}{\partial x_i \partial x_j}\right)} = 0 \tag{9.35}$$

So the first and third terms on the RHS of Eq. 9.34 cancel out.

$$2\overline{\frac{\partial}{\partial x_i}u_j' s_{ij}} - 2\nu\overline{s_{ij}\frac{\partial u_j'}{\partial x_i}} = \frac{1}{2}\frac{\partial^2}{\partial x_i \partial x_i}\overline{k^2} - \overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}} \tag{9.36}$$

Upon substitution into Eq. 8.19, we are left with:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u_i'u_i'}\right) + \bar{u}_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i'u_i'}\right) = & - \frac{\partial}{\partial x_i}\overline{u_i'\left(\frac{p'}{\rho} + \frac{1}{2}u_j'u_j'\right)} - \overline{u_i'u_j'}\frac{\partial \bar{u}_j}{\partial x_i} \\ & + \frac{1}{2}\nu\frac{\partial^2}{\partial x_i \partial x_i}\overline{k^2} - \nu\overline{\frac{\partial u_j'}{\partial x_i}\frac{\partial u_j'}{\partial x_i}}\end{aligned} \tag{9.37}$$

In most of the literature the viscous dissipation is written as

$$\epsilon = \nu \overline{\frac{\partial u'_j}{\partial x_i} \frac{\partial u'_j}{\partial x_i}} \tag{9.38}$$

This is not exactly true as can be seen by comparing Eq. 8.19 and Eq. 9.37. Note that although neither equation involves any approximations (except for incompressibility), the correct form for the dissipation rate is

$$\epsilon = 2\nu \overline{\frac{\partial u'_j}{\partial x_i} s'_{ij}} = 2\nu \overline{s'_{ij} s'_{ij}} \tag{9.39}$$

9.4.1 Closure of the diffusive transport

The diffusive transport term is usually taken to be proportional to the gradient of the turbulent kinetic energy:

$$-\overline{u'_i \left(\frac{p'}{\rho} + \frac{1}{2} u'_j u'_j \right)} = C_2 \nu_t \frac{\partial k}{\partial x_i} \tag{9.40}$$

The constant of proportionality has been written as proportional to the eddy viscosity. This constant, C_2 is determined from data and can be interpreted as a diffusion coefficient. The motivation here is that the turbulent transport acts as an enhanced diffusivity. Modeling it as such makes sense on physical grounds.

The dissipation term must also be modeled. For high Reynolds number flow we have discussed previously that the dissipation rate is approximately independent of viscosity and depends only on the kinetic energy supply rate determined by the large eddies. From dimensional reasoning we showed that $\epsilon \sim U^3/l$. We then model the dissipation term as

$$\nu \overline{2 s_{ij} \frac{\partial u'_j}{\partial x_i}} = C_3 \frac{k^{\frac{3}{2}}}{L} \tag{9.41}$$

C_3 is another empirical diffusion coefficient that must be determined by comparison with experimental data.

To complete the one-equation turbulence model formulation it is necessary to determine the length scale, L. It is the determination of L that provides the primary distinction among the various one equation models. Unfortunately, L is no easier to determine for the one equation model than for the mixing length model itself. The length scale L varies widely from flow to flow and except for thin shear layers, little data exists on its values. The following are a few references that discuss some of the ways the length scale has been determined. In practice however, the one equation models are not used very much. For more discussion on these models and the computation of the length scales two good references are Rodi[5] and Reynolds and Cebeci [6]. The most widely used models are ones in which two differential equations are solved to model the eddy viscosity. We now move on to discuss these types of models.

9.4.2 Two Equation Models

To alleviate the difficulty (and arbitrariness) of specifying the length scale, it has been suggested that the length scale be determined by a transport equation of its own. This is a reasonable approach as the length scale in a turbulent flow will be effected by many of the turbulent transport processes that affect the transport of mass, momentum, and energy. For example, if the flow is at some initial time, t_0, characterized by a length scale distribution, $L(x)$, it will at a later time have been convected to provide a different distribution. Furthermore, vortex stretching will tend to decrease length scales where this mechanisms is strong, and viscous dissipation will destroy the smallest length scales, leading to an increase in the length scale. Describing these processes by means of a differential transport equation is the goal of the two equation models.

A number of different equations have been derived that give us length scale information. Most of these equations do not solve for the length scale itself, but the length scale is easily obtainable. For example, any function of the form $Z = k^n L^m$ will give the length scale L as a modeled kinetic energy equation can be solved for k (see [5]). The equations that have been derived are mostly of the form:

$$\begin{aligned}\frac{\partial Z}{\partial t} + \bar{u}_i \frac{\partial Z}{\partial x_i} = & \quad \frac{\partial}{\partial x_j}\left(\frac{\sqrt{k}L}{\sigma_z}\right) \\ & - \quad c_{z1}\frac{Z}{k}\overline{u_i' u_j'}\frac{\partial \bar{u}_i}{\partial x_j} - c_{z2} Z \frac{\sqrt{k}}{L} + S \end{aligned} \tag{9.42}$$

The important thing about this equation is that in order to have a realistic representation of a turbulence length scale ($L = (Z/k^{-n})^{-m}$), the quantity Z that is solved for must be a property that is representative of the turbulence. In the above equation there are also a number of constants that must again be determined from data. These are represented by σ_z, c_{z1}, and c_{z2}. The first term on the RHS represents a diffusion of L, the second term a production (lengthening) of L, and the last two terms a loss in L. The last source term is represented only by S to illustrate the differences between the various forms of Z (or equivalently, the various ways of determining the length scale L).

By far the most popular of the two equation models is the $k - \epsilon$ model. This is partly a result of the ease in deriving an equation for the dissipation. From dimensional reasoning, the length scale L is determined from k and ϵ as:

$$L \propto \frac{k^{3/2}}{\epsilon} \tag{9.43}$$

Plugging this expression for L into Eq. 9.33 gives the following for the eddy viscosity:

$$\nu_t = C_\mu \frac{k^2}{\epsilon} \tag{9.44}$$

The full transport equation for the dissipation is obtained by manipulating the equation for the fluctuating momentum field. For the homogeneous dissipation, the equation can be written as :

$$\begin{aligned}
\frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} = & - 2\nu \frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{\left(\frac{\partial u_i'}{\partial x_j}\frac{\partial p'}{\partial x_j}\right)} - \nu \frac{\partial}{\partial x_k}\overline{\left(u_k' \frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_j}\right)} - 2\nu \overline{u_k' \frac{\partial u_i'}{\partial x_j}} \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_k} \\
& - 2\nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_k}} \frac{\partial \bar{u}_k}{\partial x_j} - 2\nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_k'}{\partial x_j}} \frac{\partial \bar{u}_i}{\partial x_k} \qquad (9.45) \\
& - 2\nu \overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial u_i'}{\partial x_k}\frac{\partial u_k'}{\partial x_j}} - 2\nu^2 \overline{\left(\frac{\partial^2 u_i'}{\partial x_j \partial x_k}\right)^2} + \mu \frac{\partial^2 \epsilon}{\partial x_k \partial x_k} \qquad (9.46)
\end{aligned}$$

This equation is also not closed (due to the unknown triple moments) and must be modeled in order to obtain a solution. The modeling of this equation for the dissipation is simplified by the following observations.

An order of magnitude analysis can be carried out to determine what terms in Eq. 9.45 dominate the transport for high Reynolds number flows. Representative length and velocity scales for the turbulence are the Kolmogorov scales, η and v ($\eta = (\nu^3/\epsilon)^{1/4}$, $v = (\nu \epsilon^{1/4})$). The velocity and length scales of the large scales (the energy containing eddies) are u' and L (the integral length scale). Using this, we can write:

$$u_i' u_j' \sim u'^2 \qquad (9.47)$$

$$\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k}\frac{\partial u_i'}{\partial x_k} \sim \left(\frac{v}{\eta}\right)^3 \qquad (9.48)$$

$$\overline{u_k' \left(\frac{\partial u_i'}{\partial x_j'}\right)^2} \sim u' \left(\frac{v}{\eta}\right)^2 \qquad (9.49)$$

Using relations we derived earlier, namely:

$$\frac{\eta}{L} \propto Re_l^{-3/4} \qquad (9.50)$$

and

$$\frac{v}{u'} \propto Re_L^{-1/4} \qquad (9.51)$$

where $Re_L = u'L/\nu$, we can see that for large Re, a number of terms can be dropped out. The first two terms on the RHS of Eq. 8.19 describe the turbulent diffusion transport. Of these two terms, the second term dominates over the first term. The third, fourth, fifth, and sixth terms represent the generation of dissipation rate. Order of magnitude analysis using Eqs. 9.47 - 9.49 reveals that the third fourth and fifth terms involving the gradients of the mean velocity are dominated by term six. Finally, the destruction of the dissipation by term eight is negligible compared with term seven. The reduced form of Eq. 9.45 then becomes:

$$\begin{aligned}\frac{\partial \epsilon}{\partial t} + \bar{u}_j \frac{\partial \epsilon}{\partial x_j} &= -\nu \frac{\partial}{\partial x_k} \overline{\left(u'_k \frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_j} \right)} \\ &- 2\nu \overline{\frac{\partial u'_i}{\partial x_j} \frac{\partial u'_i}{\partial x_k} \frac{\partial u'_k}{\partial x_j}} - 2\nu^2 \overline{\left(\frac{\partial^2 u'_i}{\partial x_j \partial x_k} \right)^2} \end{aligned} \tag{9.52}$$

The terms that are remaining are modeled in a fairly standard manner. The modeled equation for the dissipation is:

$$\frac{\partial \epsilon}{\partial t} + \bar{u}_i \frac{\partial \epsilon}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_\epsilon} \frac{\partial \epsilon}{\partial x_i} \right) + c_{1e} 2k \bar{S}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - c_{2e} \frac{\epsilon^2}{k} \tag{9.53}$$

The constants in these equations must be determined by fits to experimental data. The complete $k - \epsilon$ model includes the solution of Eq. 9.53 and the k equation, (8.19) who's model form is (using 9.40 and 9.41):

$$\frac{\partial k}{\partial t} + \bar{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i} \left(\frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x_i} \right) + 2\nu_t \bar{S}_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \epsilon \tag{9.54}$$

It should be noted that the modeling of the kinetic energy equation has a stronger physical justification than the equation for the dissipation. The above discussion should not be interpreted as placing too much mathematical significance on the modeled equations. In fact, it is probably best to interpret the modeled equation as the starting point for the mathematical representation as opposed to viewing the modeling as an attempt to accurately mimic each of the terms in the dissipation equation.

The model is completed by use of Eq. 9.44 to give the eddy viscosity, ν_t. From grid turbulence measurements, the constant $c_{2\epsilon}$ has been measured in the range of 1.8-2.0. $c_\mu$has been determined from measurements in shear layers to be approximately 0.09. The values for all constants in the $k - \epsilon$ model as suggested by Launder and Spalding [7] are:

$$\begin{aligned} c_\mu &= 0.09 \\ c_{1\epsilon} &= 1.44 \\ c_{2\epsilon} &= 1.92 \\ \sigma_k &= 1.0 \\ \sigma_\epsilon &= 1.3 \end{aligned} \tag{9.55}$$

The problem of determining these "constants" is not an easy one. We have stressed earlier that turbulence is not a fluid property, but a flow property that varies widely from application to application. The "constants" listed above are not generally either constant of universal. It is likely that for various applications the best choice for these "constants" has not yet been determined. Even when studying fairly simple flows, the values given in Eq. 9.55 can vary significantly. This again points to the *postdictive* nature of turbulence models: Experiments are needed to tune models before they can be applied, and then they can only be used with confidence to predict flows similar to the validating flows. Improvements to the $k - \epsilon$ models have been attained in some applications by replacing the constants in the models with functional relationships. This can yield a wider range of validity. The functional relationships, however, are themselves only empirical fits to experimental data.

Besides the somewhat suspect modeling assumptions in the dissipation equation, more fundamental limitations are present in the $k - \epsilon$ model. Eqs. 9.27 and 9.33) show that the eddy viscosity is assumed isotropic. That is, the same value is given for each of the stress components. Since the turbulent stresses can develop and interact separately, this assumption cannot be expected to be too valid. Also, the assumption of gradient transport does not always hold. In fact, we will see later some very common mixing configurations in which the mixing process appears to be quite different from that described by gradient transport. Another difficulty with the $k - \epsilon$ formulation is that it is limited to a single mixing frequency (ϵ/k) or time scale (k/ϵ). Some multiple scale models have been suggested. Again, some references will be provided at the end of this section.

In variable density flows the situation becomes even more complicated due to the fluctuating density terms and addition terms must be modeled. In combustion, for example, other generation terms due to combustion heat release and expansion must be considered. The physics here is much different than in constant density flows and should be modeled in a way that allows this physics to be described in a realistic manner. Buoyancy also can lead to turbulence production. Some references to some fairly recent papers and review articles on the modeling of both constant density and variable density turbulent flows is given at the end of this section.

9.4.3 What's Really Going On Here

Besides attempting to provide a physical understanding of what is going on in turbulent flows and repeating what is known about how turbulence effects are treated in engineering studies, another goal of this class is to provide you with the necessary understanding to intelligently apply existing models - or at least understand what the models really mean. One way to interpret the whole idea of the $k - \epsilon$ model is to realize that what it is attempting to do is provide reasonable descriptions of the diffusive nature of turbulence in terms of an eddy viscosity. This is done by deriving a form for the eddy viscosity based on some key features of the turbulence, i.e., the length, time, and velocity scales of the turbulence. Recall that the viscosity is a length times a velocity scale. If this viscosity, or "eddy" viscosity for the turbulence is formed using "turbulence" length and time scales, it means that all features of the flow smaller than these scales are really being modeled. That is, no information, say below $L = k^{3/2}/\epsilon$ is retained - it is all "smeared out" by the eddy viscosity. Now there is nothing wrong with this provided the gradient transport formulation that has been used is a reasonable representation of the physics. In many cases it is. Note that in a numerical simulation if you increase the resolution, you may get better results in terms of minimizing discretization errors, but you won't be able to explicitly simulate structure at smaller scales as everything below the $k-\epsilon$ length scale is modeled by a viscous diffusion.

9.5 Reynolds Stress Transport Equations

The problems with the eddy viscosity models could in theory be avoided by solving a transport equation for the Reynolds stresses directly. This idea was suggested back as long ago as the 1920's. The next step in sophistication of the turbulence models using Reynolds averaging techniques is in the development of a modeled equation for the Reynolds stresses.

To obtain an equation for the Reynolds stresses one can first multiply the Navier-Stokes equation for the V_i component of velocity by u_i'. Next reverse the subscripts i and j in the N-S equation and multiply by u_j'. Adding these two resulting equations together and then averaging will give the appropriate equation for the Reynolds stresses. The exact equation for the transport of the Reynolds stresses can be expressed as:

$$\frac{\partial}{\partial t}R_{ij} + \bar{u}_k \frac{\partial}{\partial x_k}R_{ij} = P_{ij} + T_{ij} - D_{ij} - \frac{\partial}{\partial x_k}J_{ijk} \tag{9.56}$$

The terms appearing in Eq. 9.56 are:

$$R_{ij} = \overline{u_i' u_j'} \tag{9.57}$$

$$P_{ij} = -\left(R_{ik}\frac{\partial \bar{u}_j}{\partial x_k} + R_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) \tag{9.58}$$

$$T_{ij} = \overline{\frac{1}{\rho}p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)} \tag{9.59}$$

$$D_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}} \tag{9.60}$$

$$J_{ijk} = -\nu \frac{\partial}{\partial x_k}R_{ij} + \overline{u_i' u_j' u_k'} + \frac{1}{\rho}\left(\overline{u_j' p'}\delta_{ik} + \overline{u_i' p'}\delta_{jk}\right) \tag{9.61}$$

R_{ij} is the Reynolds stress tensor, P_{ij} is the production term, T_{ij} is called the pressure - strain term, D_{ij} is the dissipation term, and J_{ijk} represents the flux of the Reynolds stress along the k direction. This flux is due to viscous diffusion, turbulent diffusion, and pressure. The separation of the terms into the above contributions is not unique. In particular, the decomposition of the pressure term into the pressure-strain-correlation and diffusive transport has important implications in the modeling of Eq. 9.56. It is not necessarily clear what form of Eq. 9.56 is the most conducive to modeling. Different choices could lead to very different results for the modeled equations. See Speziale [8] for other treatments.

Although the need to model the Reynolds stresses has been eliminated in this approach, the task of modeling the other terms that appear in Eq. 9.56 has greatly increased. In fact, the terms represented by Eqs. 9.59 - 9.61 must all be modeled.

The pressure-strain term has no counterpart in the kinetic energy equation. The tendency of this term is to enhance isotropy. It has no effect on the total energy balance, as it only *redistributes* energy among its various components. The name "pressure-strain" arises because this term is a correlation between the pressure and the fluctuating strain rate tensor. Before we discuss the modeling of this crucial term, let us first observe the other terms that must be modeled.

The belief that the dissipation for high Reynolds number flow is isotropic leads to

$$D_{ij} = \frac{2}{3}\epsilon\delta_{ij} \tag{9.62}$$

where ϵ is the isotropic dissipation discussed extensively earlier. The flux term, J_{ijk} is usually modeled using a gradient transport assumption. One such form was proposed by Daly and Harlow [9]:

$$J_{ijk} = c_s \frac{\partial}{\partial x_l}\left(\frac{k}{\epsilon}\overline{u'_k u'_l}\frac{\partial \overline{u'_i u'_j}}{\partial x_k}\right) \tag{9.63}$$

A more general model is given by Hanjalic and Launder (1972) and Launder et al. [10] is:

$$J_{ijk} = C\frac{k}{\epsilon}\left(R_{il}\frac{\partial}{\partial x_l}R_{jk} + R_{jl}\frac{\partial}{\partial x_l}R_{ik} + R_{kl}\frac{\partial}{\partial x_l}R_{ij}\right) \tag{9.64}$$

See also Lumley [11].

The pressure-strain term is unique to the Reynolds stress equations and its accurate specification is crucial to the overall performance of the modeling. Some insight into modeling the pressure-strain redistribution term can be obtained by looking into the exact equation for the pressure fluctuation. Taking the divergence of the full momentum equation, and then separating the variables into their mean and fluctuating components. The result of this operation (assuming an incompressible fluid and neglecting gravity forces) is:

$$\frac{1}{\rho}\frac{\partial^2}{\partial x_i \partial x_i}p' = -\left(\frac{\partial u'_i}{\partial x_j}\frac{\partial u'_j}{\partial x_i} + \frac{\partial u'_i}{\partial x_j}\frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{9.65}$$

This Poisson equation for the pressure has two terms on the RHS that will effect the pressure field. The first only contains turbulence terms, the second term also contains the mean deformation tensor. Each of these two contributions to the pressure-strain correlation is usually modeled separately. In the absence of any mean shear, the pressure-strain terms acts to returns an anisotropic flow to isotropic. The "return to isotropy" contribution has been modeled by Rotta (1972)(and discussed by Gibson et al. [12]; Rodi [5]) as:

$$T^1_{ij} = -c_1\frac{\epsilon}{k}\left(\overline{u'_i u'_j} - \frac{2}{3}\delta_{ij}k\right) \tag{9.66}$$

where the superscript 1 has been used to indicate that this is the first part of the pressure-strain correlation.

The other contribution to the pressure-strain correlation is called the "mean shear" or "rapid" part. This term has been modeled by Launder et al. [10] as:

$$T_{ij}^2 = -\frac{c_2+8}{11}\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right) - \frac{30c_2-2}{55}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right)k - \frac{8c_2-2}{11}\left(D_{ij} - \frac{2}{3}\delta_{ij}P\right) \quad (9.67)$$

where

$$D_{ij} = -\left(\overline{u_i'u_k'}\frac{\partial \bar{u}_k}{\partial x_j} + \overline{u_j'u_k'}\frac{\partial \bar{u}_k}{\partial x_i}\right) \quad (9.68)$$

P is the kinetic energy production rate as given by Eqs. 8.18 and 8.19.

A simpler more widely used model is a simplification of the above model:

$$T_{ij}^2 = -c_2'\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right) \quad (9.69)$$

This model for the mean shear contribution to the pressure-strain correlation has provided a reasonable approximation as the first term in Eq. 9.68 was shown by Launder et al.[10] to make the major contribution. In summary, the complete pressure-strain model can be expressed as

$$\begin{aligned} T_{ij} = & - c_1\frac{\epsilon}{k}\left(\overline{u_i'u_j'} - \frac{2}{3}\delta_{ij}k\right) \\ & - c_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P\right) \end{aligned} \quad (9.70)$$

In general, there is also a contribution due to buoyancy. This is described in [12]. A more general pressure-strain model can be found in Jones and Musogne [13]

The above is a very abbreviated introduction into the closure of the Reynolds stress equations. A set of notes prepared by Gibson [12] for a short course on turbulence modeling contains a relatively recent and up to date collection of references on Reynolds transport modeling. It is a good source for those wishing to use turbulence models in their work. One of the biggest shortcomings of Reynolds stress transport modeling is the lack of data, or the inability to obtain data to study and evaluate the performance of the proposed models for the various terms. Many of the higher order correlations, especially those involving pressure perturbations are very difficult to measure.

9.6 Recapitulation

It's clear from the above discussions that the modeling of turbulent flows is a difficult task. There also is no general agreement concerning the best turbulence models to use for a particular application. The use of turbulence models is the only way in which flows of practical engineering application can be solved. This, as we have pointed out many times, is due to the extreme range of time and space scales that exist in turbulent flows.

The type of modeling we have discussed in this section is all based upon the Reynolds averaged form of the governing equations for mass, momentum, kinetic energy, dissipation, and the Reynolds stresses. A difficulty with this approach is that this "global" type of averaging procedure results in the loss of information about *all* scales of the flow. One of the effects of this is the closure problem. In our discussions of the various length scale regimes in turbulent flows we have shown that the small scale structure of turbulent flows can be expected to behave in a relatively universal way. The large scale motions, on the other hand are highly anisotropic, and their development is highly dependent on the operating conditions and geometry of the flow. Furthermore, it is the large scale motions that are responsible for the major contribution to the turbulent transport of mass, momentum, and energy. It is very unlikely then that a universal model can be expected to reproduce the large scale structure for a wide class of flows.

We discussed two basic approaches to modeling the Reynolds stresses that appear in the averaged momentum equation. One based on introducing an eddy viscosity to model the diffusive properties of turbulence, and another based on the solution of the transport equations for the Reynolds stresses. The eddy viscosity models include three basic types: Zero equation, one equation, and two equation models. The zero equation models require the specification of a velocity scale and a mixing length scale. The one equation models solve a transport equation for the velocity scale, generally the turbulent kinetic energy. The specification of a length scale is still required. The two equation models include an additional transport equation to give the mixing length information. The most popular two equation model is the $k\epsilon$ model which includes a transport equation for both the kinetic energy and the dissipation. This is probably the most widely used turbulence model in use today. These equations are themselves unclosed and modeling assumptions must be made to achieve their solution. From a comparison with the k and ϵ equation in their exact and modeled form, it is seen that the modeled dissipation equation appears to be on a shakier foundation than the kinetic energy equation.

A further degree of sophistication is achieved by solving for the Reynolds stresses by their own transport equations. These equations also are not closed due to the appearance of pressure-strain correlation and triple moments of the fluctuating velocities. This is debatably considered the highest degree of complexity that can be realistically used in practical applications. Although with the increasing capabilities of computing power and the continued development of large eddy simulation (LES) and refinement of subgrid models, more detailed approaches to simulating turbulent flow are gaining popularity. (LES will be discussed later in the course.) The addition of the Reynolds transport equations implies that much more computing time is necessary due to the additional transport equations that must be solved. Unfortunately, the uncertainty in some of the modeling assumptions detracts from the advantages of solving for the Reynolds stresses directly. For many complex flows, however, the lower order models are fundamentally incapable of reproducing the correct behavior. In such cases it should be expected that better performance could be obtained by the modeled Reynolds stress equations. Unfortunately, these models are expensive to evaluate and are still under development. As a result, most engineering calculations of turbulent flows use the $k - \epsilon$ model. All these models are far from perfect. Much more work needs to be done before the Reynolds stress models will consistently outperform the lower order $k - \epsilon$ models.

It must be emphasized that numerical calculations employing turbulence models of the type discussed in this section do not directly enhance our fundamental understanding of turbulence. Eventually, they all reduce to a correlation of with experimental data at some point. A well developed turbulence model must contain a high degree of physical reality in its formulation. The developer of a turbulence model attempts to capture the physics in a simplified representation. Any results obtained from a turbulence model therefore only describe the physics built into it, and cannot be used to identify physical mechanisms directly. Indirectly, however, comparison of model performance with real physical data can point to shortcomings in the models. Furthermore, most turbulence models are developed with a conscientious effort towards understanding the appropriate physics and how to best model it. Physical understanding will therefore be a bi-product of the search for a more universal model.

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