More Geometry for Graphics

January 12, 2007

Review from last time

- Vector spaces
- Points
- Vectors
- Affine Transformations
- Implementation tips

Implementation notes

- Dot product and cross product are awkward. I have done it two ways in C++:
  - As a standalone function: Dot(A, B)
  - As a member method: A.dot(B)

- I don't recommend overloading ^ or some other obscure operator. Operator precedence will bite you and nobody will be able to read your code

Other useful vector operations

- The following operations will also be be useful:
  - length (or magnitude)
  - length squared (length2)
  - normalize

The net result

- I don't care if you only vaguely recall/ understand the stuff about spaces, but understand and remember the properties!
- These properties can be very useful in optimizing programs and deriving equations. Know them cold - it could make the difference between having fun in this class and struggling all semester!

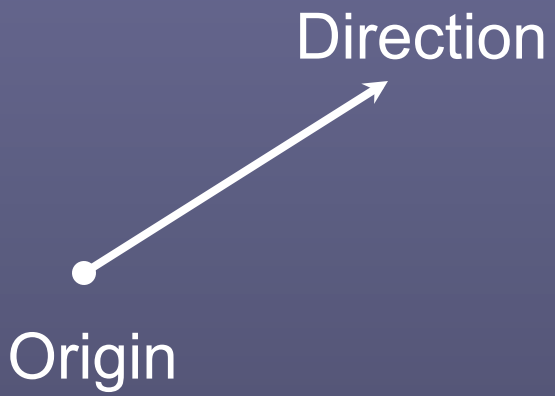
Geometric entities

- Now we can finally define some geometric entities:
- Line segment: two points
- Ray: a point and a vector
- Line: either of the above

Rays

- A ray consists of a point and a vector:

```
Class Ray {
 Point origin;
 Vector direction;
…
};
```

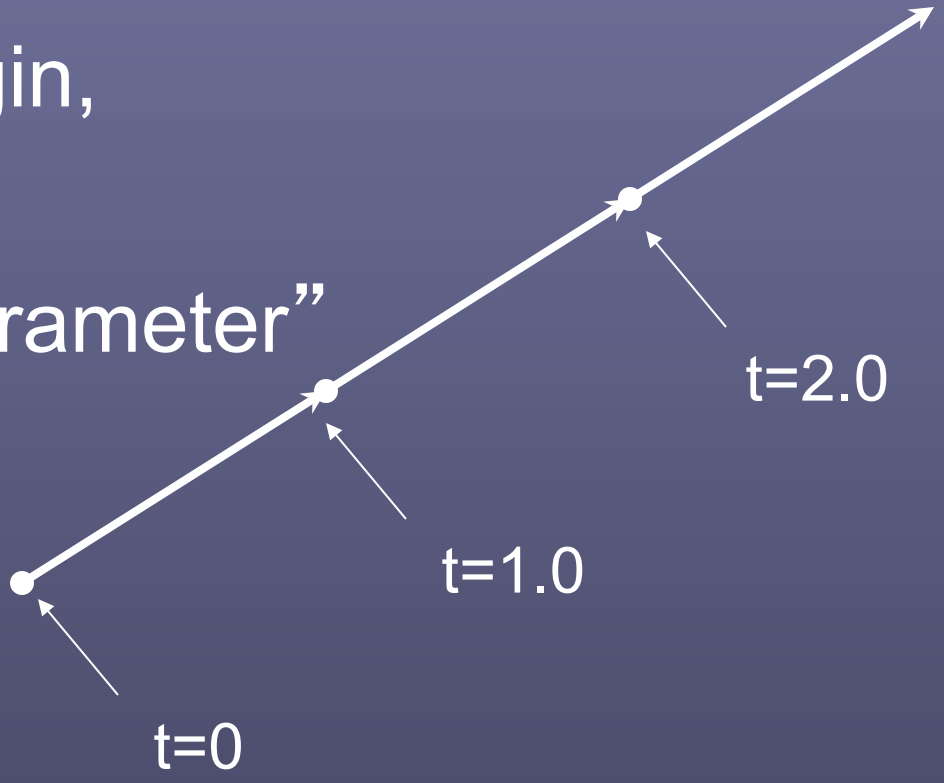


Parametric Rays

- We usually parameterize rays:

Where O is the origin,
V is direction,
and t is the "ray parameter"

$$\vec{P} = \vec{O} + t\vec{V}$$



Planes

- The equation for a plane is: $ax + by + cz + d = 0$
- A plane can be defined with a Vector (the normal to the plane) and a point on the plane:

$$a = N_x; b = N_y; c = N_z$$

$$d = -\vec{N} \bullet \vec{P}$$

- Alternative form of plane equation:

$$\vec{N} \bullet \vec{P} + d = 0$$

Plane properties

- “Plugging in” a point to the plane equation yields a scalar value:

0: Point is on the plane

+: on the normal side of the plane

-: opposite the normal side of the plane

Multiplying a,b,c,d by a (strictly) positive number yields the same plane

Multiplying a,b,c,d by a (strictly) negative number flips the normal

a==b==c==0 is a degenerate plane

Colors

- For the purpose of this class, Color is Red, Green, Blue
- Range is 0-1
- Other color models will be discussed briefly in a few weeks
- Colors should be represented using the “float” datatype - others just don’t make sense
- Define operators that make sense

Implementation notes

- Implement Color * Color, Color * scalar, Color - Color, Color + Color
- Don’t go overboard with other operations - you may never use them and by leaving them missing you may avoid shooting yourself in the foot

Images

- Images are just a 2D array of Pixels
- Pixels are not Colors - they can be lighter weight (3 chars)
- Implement a way to set a pixel from a color (scale and clamp to 0-255)
- Implement a way to write out images

Image formats

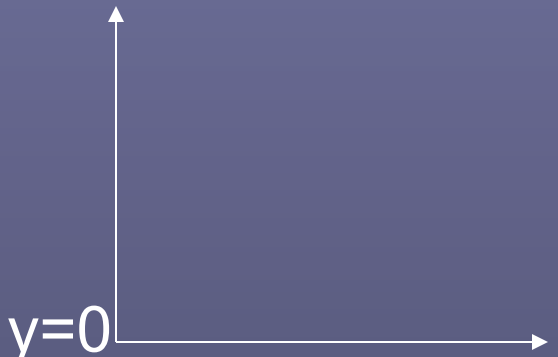
- Select an image format to use to write out images. I recommend PPM:

http://netpbm.sourceforge.net/doc/ppm.html

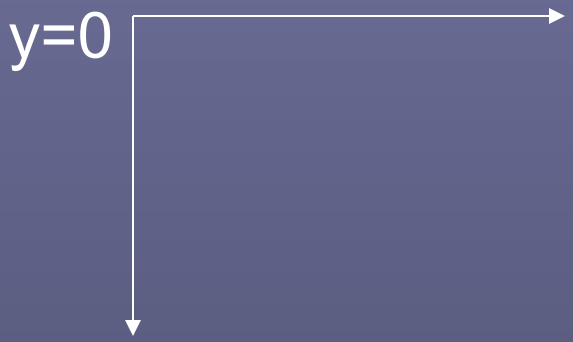
- There are several free image viewers for PPM for all platforms
- You will need to convert them to PNG or JPEG for your web page
- You may want to write a mechanism to display them directly using OpenGL (glDrawPixels)
- You are also welcome to use a library to write out images in PNG, JPEG, or another format

Image gotchas

- Be careful - image coordinate system is “upside down”



Real world
Our ray tracer
OpenGL
Taught since 2nd grade



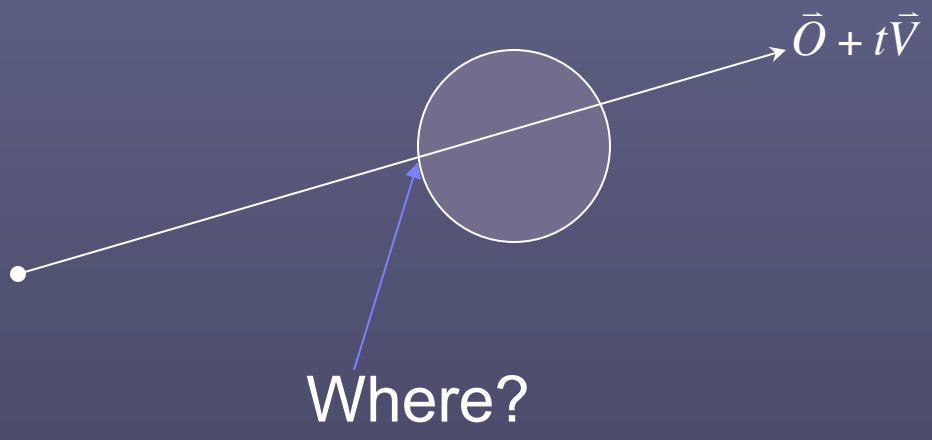
Televisions
Raster Images
Other 1950’s technology

Geometric Queries

- Back to the original question:

  What queries can we perform on our virtual geometry?

- Ray tracing: determine if (and where) rays hit an object



Ray-plane intersection

- To find the intersection of a ray with a plane, determine where both equations are satisfied at the same time:

$$\vec{N} \bullet \vec{P} + d = 0 \text{ and } \vec{P} = \vec{O} + t\vec{V}$$

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$$\vec{N} \bullet \vec{O} + t\vec{N} \bullet \vec{V} + d = 0$$

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$$\vec{N} \bullet \vec{O} + t\vec{N} \bullet \vec{V} + d = 0$$

$$t\vec{N} \bullet \vec{V} = -\left(d + \vec{N} \bullet \vec{O} \right)$$

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$$t\vec{N} \bullet \vec{V} = -\left(d + \vec{N} \bullet \vec{O} \right)$$

$$t = -\frac{\left(d + \vec{N} \bullet \vec{O} \right)}{\vec{N} \bullet \vec{V}}$$

Ray-plane intersection

- If $\vec{N} \bullet \vec{V} = 0$ (or close to it) then the ray is parallel to the plane
- The parameter t defines the point where the ray intersects the plane
- To determine the point of intersection, just plug t back into the ray equation $\left(\vec{O} + t\vec{V}\right)$

Ray-sphere intersection

- We can do the same thing for other objects
- What is the implicit equation for a sphere centered at the origin?

Ray-sphere intersection

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- What is the implicit equation for a sphere centered at the origin?

$$x^2 + y^2 + z^2 - r^2 + 0$$

Ray-sphere intersection

Sphere: $x^2 + y^2 + z^2 - r^2 = 0$

Ray: $\left[O_x + tV_x, O_y + tV_y, O_z + tV_z\right]$

Ray-sphere intersection

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$$\left(O_x + tV_x\right)^2 + \left(O_y + tV_y\right)^2 + \left(O_z + tV_z\right)^2 - r^2 = 0$$

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$$O_x^{\,2} + 2tV_x + t^2V_x^{\,2} + O_y^{\,2} + 2tV_y + t^2V_y^{\,2} + O_z^{\,2} + 2tV_z + t^2V_z^{\,2} - r^2 = 0$$

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$$t^2\left(V_x^{\,2} + V_y^{\,2} + V_z^{\,2}\right) + 2t\left(V_x + V_y + V_z\right) + O_x^{\,2} + O_y^{\,2} + O_z^{\,2} - r^2 = 0$$

Ray-sphere intersection

$$t^2\left(V_x^2 + V_y^2 + V_z^2\right) + 2t\left(V_x + V_y + V_z\right) + O_x^2 + O_y^2 + O_z^2 - r^2 = 0$$

A quadratic equation, with

$$a = V_x^2 + V_y^2 + V_z^2$$

$$b = 2\left(V_x + V_y + V_z\right)$$

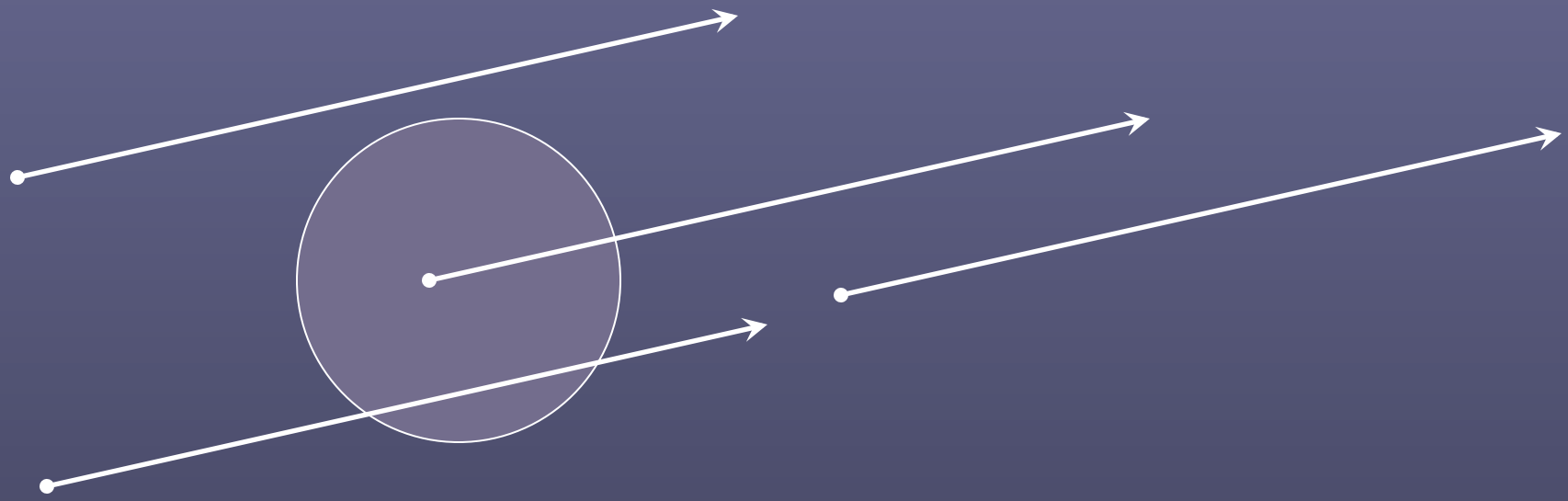
$$c = O_x^2 + O_y^2 + O_z^2 - r^2$$

roots:

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Ray-sphere intersection

- If the discriminant $\left(b^2 - 4ac\right)$ is negative, the ray misses the sphere
- Otherwise, there are two distinct intersection points (the two roots)



Ray-sphere intersection

- What about spheres not at the origin?
- For center C, the equation is:

$$(x - C_x)^2 + (y - C_y)^2 + (z - C_z)^2 - r^2 = 0$$

- We could work this out, but there must be an easier way…

Ray-sphere intersection, improved

- Points on a sphere are equidistant from the center of the sphere
- Our measure of distance: dot product
- Equation for sphere:

$$\left(\vec{P} - \vec{C}\right) \bullet \left(\vec{P} - \vec{C}\right) - r^2 = 0$$

Ray-sphere intersection, improved

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$$\vec{P} = \vec{O} + t\vec{V}$$

$$\left(\vec{O} = t\vec{V} - \vec{C}\right)\bullet\left(\vec{O} = t\vec{V} - \vec{C}\right) - r^2 = 0$$

$$t^2\vec{V}\bullet\vec{V} + 2t\left(\vec{O} - \vec{C}\right)\bullet\vec{V} + \left(\vec{O} - \vec{C}\right)\bullet\left(\vec{O} - \vec{C}\right) - r^2 = 0$$

Ray-sphere intersection, improved

$$t^2\vec{V}\bullet\vec{V} + 2t\left(\vec{O}-\vec{C}\right)\bullet\vec{V} + \left(\vec{O}-\vec{C}\right)\bullet\left(\vec{O}-\vec{C}\right) - r^2 = 0$$

Vector $\vec{O}' = \vec{O} - \vec{C}$

$a = \vec{V}\bullet\vec{V}$

$b = 2\vec{O}'\bullet\vec{V}$

$c = \vec{O}'\bullet\vec{O}' - r^2$

Solve for the roots the same way

There are still ways that we can improve this - next week