More Geometry for Graphics

January 12, 2007
Review from last time

- Vector spaces
- Points
- Vectors
- Affine Transformations
- Implementation tips
Implementation notes

• Dot product and cross product are awkward. I have done it two ways in C++:
  – As a standalone function: `Dot(A, B)`
  – As a member method: `A.dot(B)`

• I don’t recommend overloading `^` or some other obscure operator. Operator precedence will bite you and nobody will be able to read your code.
Other useful vector operations

- The following operations will also be useful:
  - length (or magnitude)
  - length squared (length2)
  - normalize
The net result

• I don’t care if you only vaguely recall/understand the stuff about spaces, but understand and remember the properties!

• These properties can be very useful in optimizing programs and deriving equations. Know them cold - it could make the difference between having fun in this class and struggling all semester!
Geometric entities

• Now we can finally define some geometric entities:
  • Line segment: two points
  • Ray: a point and a vector
  • Line: either of the above
Rays

• A ray consists of a point and a vector:

```java
Class Ray {
    Point origin;
    Vector direction;
    ...
};
```
Parametric Rays

- We usually parameterize rays:

Where \( O \) is the origin, \( V \) is direction, and \( t \) is the “ray parameter”

\[
\vec{P} = \vec{O} + t\vec{V}
\]

\( \vec{P} \) = \( \vec{O} \) + \( t \vec{V} \)
Planes

• The equation for a plane is: \( ax + by + cz + d = 0 \)
• A plane can be defined with a Vector (the normal to the plane) and a point on the plane:

\[
\begin{align*}
  a &= N_x; b = N_y; c = N_z \\
  d &= -\vec{N} \cdot \vec{P}
\end{align*}
\]
• Alternative form of plane equation:

\[
\vec{N} \cdot \vec{P} + d = 0
\]
Plane properties

• “Plugging in” a point to the plane equation yields a scalar value:
  0: Point is on the plane
  +: on the normal side of the plane
  -: opposite the normal side of the plane

Multiplying \(a,b,c,d\) by a (strictly) positive number yields the same plane
Multiplying \(a,b,c,d\) by a (strictly) negative number flips the normal
\(a==b==c==0\) is a degenerate plane
Colors

- For the purpose of this class, Color is Red, Green, Blue
- Range is 0-1
- Other color models will be discussed briefly in a few weeks
- Colors should be represented using the “float” datatype - others just don’t make sense
- Define operators that make sense
Implementation notes

• Implement Color * Color, Color * scalar, Color - Color, Color + Color

• Don’t go overboard with other operations - you may never use them and by leaving them missing you may avoid shooting yourself in the foot
Images

- Images are just a 2D array of Pixels
- Pixels are not Colors - they can be lighter weight (3 chars)
- Implement a way to set a pixel from a color (scale and clamp to 0-255)
- Implement a way to write out images
Image formats

• Select an image format to use to write out images. I recommend PPM:
  http://netpbm.sourceforge.net/doc/ppm.html
• There are several free image viewers for PPM for all platforms
• You will need to convert them to PNG or JPEG for your web page
• You may want to write a mechanism to display them directly using OpenGL (glDrawPixels)
• You are also welcome to use a library to write out images in PNG, JPEG, or another format
Image gotchas

- Be careful - image coordinate system is “upside down”

Real world
- Our ray tracer
- OpenGL
- Taught since 2nd grade

Televisions
- Raster Images
- Other 1950’s technology
Geometric Queries

• Back to the original question:
  What queries can we perform on our virtual geometry?

• Ray tracing: determine if (and where) rays hit an object
Ray-plane intersection

• To find the intersection of a ray with a plane, determine where both equations are satisfied at the same time:

\[ \vec{N} \cdot \vec{P} + d = 0 \quad \text{and} \quad \vec{P} = \vec{O} + t\vec{V} \]
Ray-plane intersection

• To find the intersection of a ray with a plane, determine where both equations are satisfied at the same time:

\[ \vec{N} \cdot \vec{P} + d = 0 \quad \text{and} \quad \vec{P} = \vec{O} + t\vec{V} \]

\[ \vec{N} \cdot (\vec{O} + t\vec{V}) + d = 0 \]
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\[ \mathbf{N} \cdot \mathbf{O} + t\mathbf{N} \cdot \mathbf{V} + d = 0 \]
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\[ \vec{N} \cdot \vec{O} + t \vec{N} \cdot \vec{V} + d = 0 \]

\[ t \vec{N} \cdot \vec{V} = -\left( d + \vec{N} \cdot \vec{O} \right) \]
Ray-plane intersection

- To find the intersection of a ray with a plane, determine where both equations are satisfied at the same time:

\[ \vec{N} \cdot \vec{P} + d = 0 \text{ and } \vec{P} = \vec{O} + t\vec{V} \]

\[ \vec{N} \cdot (\vec{O} + t\vec{V}) + d = 0 \]

\[ \vec{N} \cdot \vec{O} + t\vec{N} \cdot \vec{V} + d = 0 \]

\[ t\vec{N} \cdot \vec{V} = -(d + \vec{N} \cdot \vec{O}) \]

\[ t = -\frac{(d + \vec{N} \cdot \vec{O})}{\vec{N} \cdot \vec{V}} \]
Ray-plane intersection

• If $\vec{N} \cdot \vec{V} = 0$ (or close to it) then the ray is parallel to the plane
• The parameter $t$ defines the point where the ray intersects the plane
• To determine the point of intersection, just plug $t$ back into the ray equation \((\vec{O} + t\vec{V})\)
Ray-sphere intersection

- We can do the same thing for other objects
- What is the implicit equation for a sphere centered at the origin?
Ray-sphere intersection

• We can do the same thing for other objects

• What is the implicit equation for a sphere centered at the origin?
  \[ x^2 + y^2 + z^2 - r^2 + 0 \]
Ray-sphere intersection

Sphere: \( x^2 + y^2 + z^2 - r^2 = 0 \)

Ray: \( [O_x + tV_x, O_y + tV_y, O_z + tV_z] \)
Ray-sphere intersection

Sphere: \( x^2 + y^2 + z^2 - r^2 = 0 \)

Ray: \( \left[ O_x + tV_x, O_y + tV_y, O_z + tV_z \right] \)

\[
\left( O_x + tV_x \right)^2 + \left( O_y + tV_y \right)^2 + \left( O_z + tV_z \right)^2 - r^2 = 0
\]
Ray-sphere intersection

Sphere: \( x^2 + y^2 + z^2 - r^2 = 0 \)

Ray: \[ [O_x + tV_x, O_y + tV_y, O_z + tV_z] \]

\[
\left( O_x + tV_x \right)^2 + \left( O_y + tV_y \right)^2 + \left( O_z + tV_z \right)^2 - r^2 = 0
\]

\[
O_x^2 + 2tV_x + t^2V_x^2 + O_y^2 + 2tV_y + t^2V_y^2 + O_z^2 + 2tV_z + t^2V_z^2 - r^2 = 0
\]
Ray-sphere intersection

Sphere: \( x^2 + y^2 + z^2 - r^2 = 0 \)

Ray: \[ [O_x + tV_x, O_y + tV_y, O_z + tV_z] \]

\[ (O_x + tV_x)^2 + (O_y + tV_y)^2 + (O_z + tV_z)^2 - r^2 = 0 \]

\[ O_x^2 + 2tV_x + t^2V_x^2 + O_y^2 + 2tV_y + t^2V_y^2 + O_z^2 + 2tV_z + t^2V_z^2 - r^2 = 0 \]

\[ O_x^2 + O_y^2 + O_z^2 + 2tV_x + 2tV_y + 2tV_z + t^2V_x^2 + t^2V_y^2 + t^2V_z^2 - r^2 = 0 \]
Ray-sphere intersection

Sphere:  \( x^2 + y^2 + z^2 - r^2 = 0 \)

Ray: \[ O_x + tV_x, O_y + tV_y, O_z + tV_z \]

\[
\left( O_x + tV_x \right)^2 + \left( O_y + tV_y \right)^2 + \left( O_z + tV_z \right)^2 - r^2 = 0
\]

\[
O_x^2 + 2tV_x + t^2V_x^2 + O_y^2 + 2tV_y + t^2V_y^2 + O_z^2 + 2tV_z + t^2V_z^2 - r^2 = 0
\]

\[
O_x^2 + O_y^2 + O_z^2 + 2tV_x + 2tV_y + 2tV_z + t^2V_x^2 + t^2V_y^2 + t^2V_z^2 - r^2 = 0
\]

\[
t^2 \left( V_x^2 + V_y^2 + V_z^2 \right) + 2t \left( V_x + V_y + V_z \right) + O_x^2 + O_y^2 + O_z^2 - r^2 = 0
\]
Ray-sphere intersection

\[ t^2 \left( V_x^2 + V_y^2 + V_z^2 \right) + 2t \left( V_x + V_y + V_z \right) + O_x^2 + O_y^2 + O_z^2 - r^2 = 0 \]

A quadratic equation, with

\[ a = V_x^2 + V_y^2 + V_z^2 \]
\[ b = 2 \left( V_x + V_y + V_z \right) \]
\[ c = O_x^2 + O_y^2 + O_z^2 - r^2 \]

roots:

\[ -b + \sqrt{b^2 - 4ac} \]
\[ -b - \sqrt{b^2 - 4ac} \]
\[ \frac{2a}{2a} \]
Ray-sphere intersection

• If the discriminant \((b^2 - 4ac)\) is negative, the ray misses the sphere
• Otherwise, there are two distinct intersection points (the two roots)
Ray-sphere intersection

• What about spheres not at the origin?
• For center $C$, the equation is:

$$\left(x - C_x\right)^2 + \left(y - C_y\right)^2 + \left(z - C_z\right)^2 - r^2 = 0$$

• We could work this out, but there must be an easier way…
Ray-sphere intersection, improved

- Points on a sphere are equidistant from the center of the sphere
- Our measure of distance: dot product
- Equation for sphere:
  \[
  (\mathbf{P} - \mathbf{C}) \cdot (\mathbf{P} - \mathbf{C}) - r^2 = 0
  \]
Ray-sphere intersection, improved

- Points on a sphere are equidistant from the center of the sphere
- Our measure of distance: dot product
- Equation for sphere:
  \[(\vec{P} - \vec{C}) \cdot (\vec{P} - \vec{C}) - r^2 = 0\]
  
  \[
  \vec{P} = \vec{O} + t\vec{V}
  \]

  \[
  (\vec{O} = t\vec{V} - \vec{C}) \cdot (\vec{O} = t\vec{V} - \vec{C}) - r^2 = 0
  \]

  \[
  t^2\vec{V} \cdot \vec{V} + 2t (\vec{O} - \vec{C}) \cdot \vec{V} + (\vec{O} - \vec{C}) \cdot (\vec{O} - \vec{C}) - r^2 = 0
  \]
Ray-sphere intersection, improved

\[ t^2 \mathbf{V} \cdot \mathbf{V} + 2t (\mathbf{O} - \mathbf{C}) \cdot \mathbf{V} + (\mathbf{O} - \mathbf{C}) \cdot (\mathbf{O} - \mathbf{C}) - r^2 = 0 \]

Vector \( \mathbf{O}' = \mathbf{O} - \mathbf{C} \)

\[ a = \mathbf{V} \cdot \mathbf{V} \]

\[ b = 2 \mathbf{O}' \cdot \mathbf{V} \]

\[ c = \mathbf{O}' \cdot \mathbf{O}' - r^2 \]

Solve for the roots the same way

There are still ways that we can improve this - next week