CS 6640: IMAGE PROCESSING
Fall 2009
Take Home Test

Name: _________________________________
Student ID Number: ____________________________

Rules:

• You make use the book. But you should work by yourself. You should write at the bottom of this page “I did this work on my own”, and you should sign it.

• No calculators.

Hints:

• The term “describe” does not mean complete sentences and paragraphs or essays. If it’s easier you may use simple bullets and meaningful phrases to answer such questions.

• If you split answers across pages (or on the backs of pages) make a clear note on the page where the question is posed to indicate you have done so. Clearly note the question number (and part) on the separate page. Please do not add pages to this for your answers (i.e. we will only grade the pages in this document).

• There are four questions for a total of 100 points. Point values are roughly correlated with the amount of time you should devote to each question. Questions will be graded as “correct” if it is mostly correct or “incorrect” if it is not mostly correct.
1. [20 pts.] This is a question about modeling and analyzing blurring in images using the Fourier transform. When a camera takes a picture the shutter opens for a short period of time and then shuts again. During this time the film or sensor effectively integrates the light falling on the imaging plane. If the camera moves while the shutter is open, the motion produces a kind of accumulation of different images at different positions, resulting in a blurry image. This is called motion blurring.

If we assume the camera moves in such a way that the sequence of images forms a straight line, the effects of this distortion (blur) can be approximated as a convolution with a 1D box (rectangle) filter. For instance, if the camera moves so that the image translates one pixel in the \( x \) direction while the shutter is open for one unit of time, the corresponding blur might be a convolution with \( h(x, y) = \text{rect}(x)\delta(y) \) where

\[
\text{rect}(x) = \begin{cases} 
1 & -1/2 < x < 1/2 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Give the equation (full analytical expression) for motion blurring in the Fourier domain for an image \( f(x, y) \).

(b) Suppose we have the same situation as above, but we leave the shutter open for \( T \) units of time (instead of one unit). What is the corresponding kernel in the space and frequency domains?

(c) Suppose the camera moves so that the images effectively translate in some direction with angle \( \theta \) (relative to the \( x \) axis of the image) (see figure below). Give the equation for corresponding \( h(x, y) \).

(d) Give the Fourier transform of \( h(x, y) \) in this case. What pattern do the zero crossings (places where the power spectrum is zero) of this function have (draw them in \((u, v)\) space)?

(e) Suppose we are given a motion-blurred image but don’t know \( T \) or \( \theta \). In general terms, what strategy might we use to try to infer \( T \) and \( \theta \). (Be brief).

(f) If we knew the duration and direction of this blurring, how could we recover the image we should have gotten without blurring? What problems would we expect to encounter?
2. [20 pts.] This question deals with the issue of building continuous interpolations of discretely sampled, greyscale images.

(a) Consider linear interpolation in one dimension, in which we wish to represent the continuous function \( f(x) \) using discrete samples \( f_i \). This interpolation process can be described as a convolution.
   - Give the mathematical equations for this (1D interpolation by convolution).
   - Give the kernel associated with linear interpolation.

(b) Give the mathematical equations for 2D interpolation by convolution, in which we would construct \( f(x, y) \) from \( f_{ij} \).

(c) Give a mathematical expression for the kernel associated with bilinear interpolation.

(d) Give a mathematical expression for 2D kernel associated with nearest-neighbor interpolation (in which we assign a value to \( (x, y) \) based on the closest discrete point \( (i, j) \)).

(e) Describe (using diagrams) the frequency characteristics of nearest-neighbor versus bilinear interpolation and explain how these would affect the resulting interpolated images. (Be brief. Use bullet points. This discussion should probably be less than 1/2 page — not too long).
3. [20 pts.] Consider the 1D function which is a cosine function of frequency $k$ modulated by a tent or triangle function of width $2a$. The resulting function is

$$f(x) = \begin{cases} 
\frac{1}{a}(1 + \frac{x}{a}) \cos(2\pi kx) & -a < x < 0 \\
\frac{1}{a}(1 - \frac{x}{a}) \cos(2\pi kx) & 0 < x < a \\
0 & \text{otherwise}
\end{cases}$$

(a) Graph this function for $a = 1$, $k = 1$.
(b) Give the Fourier transform of this function, $F(s)$.
(c) For certain kinds of filters, it’s convenient of the total integral of the function is zero. I.e. $\int_{-\infty}^{\infty} f(x)dx = 0$. What is the general relationship between $k$ and $a$ that ensures this?
(d) What kind of filter is this (LP, BP, HP)? Say why.
4. [20 pts.] Consider the image alignment or warping problem based on correspondences 
\((\bar{c}_1, \bar{c}_1'), (\bar{c}_2, \bar{c}_2'), \ldots (\bar{c}_N, \bar{c}_N')\), where \(\bar{c}_i = (x_i, y_i)\) and \(\bar{c}_i' = (x_i', y_i')\). (Hint: Don’t generate a lot of text for this question. Give short answers and equations.)

(a) Suppose we wish to find a simple translation \((x_0, y_0)\), and we wish to overconstrain the system. What penalty function would we minimize (to get a good answer) and what linear system would we use to solve the problem? Give the equations/matrices.

(b) Suppose we wish to find both a translation and an isotropic scaling (same in all directions) of one of the images. What linear system would we use? Give equations/matrices.

(c) Suppose we wish to find a rotation between two images that best describes these correspondences. The problem is no longer linear. Give the equation for the penalty function and describe (briefly) one strategy for finding the best rotation.
5. [20 pts] Consider a Gaussian in 2D with standard deviation $\sigma$, which is:

$$g_\sigma(x, y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}.$$

(a) Prove (through equations) that the convolution of two Gaussians of different widths (standard deviations) is another Gaussian. I.e.

$$g_{\sigma_1}(x, y) \otimes g_{\sigma_2}(x, y) = g_{\sigma_3}(x, y)$$

(b) What is the expression for $\sigma_3$ in terms of $\sigma_1$ and $\sigma_2$?