Discrete Sampling and Aliasing

- Digital signals and images are discrete representations of the real world
  - Which is continuous
- What happens to signals/images when we sample them?
  - Can we quantify the effects?
  - Can we understand the artifacts and can we limit them?
  - Can we reconstruct the original image from the discrete data?

Digital signals and images come from the world—real life, which is continuous by nature.

Continuous functions have to be converted into a sequence of discrete values before they can be processed by a computer.

This is achieved by
- sampling → the domain
- quantization → the range

E.g.

By getting equally spaced samples from a continuous function.

\[ f_k = f(k\Delta T) \]
A Mathematical Model of Discrete Samples

Impulse and their Sifting Property

* A unit impulse of a continuous variable $t$ located at $t = 0$:
  \[ s(t) = \begin{cases} \infty & t = 0 \\ 0 & \text{o.w.} \end{cases} \]
  where $\int s(t) \, dt = 1$

* Sifting Property w.r.t. integration
  \[ \int_{-\infty}^{\infty} f(t) \, s(t) \, dt = f(0) \]

Integration with a function yields the value of that function at the location of the impulse.

In general:

\[ \int_{-\infty}^{\infty} f(t) \, s(t-t_0) \, dt = f(t_0) \]

The impulse train (Shah functional):

It is periodic with period $\Delta T$,
\[ \text{i.e. in even } \Delta T, \text{ we will have an impulse.} \]
A Mathematical Model of Discrete Samples

- **Goal**
  - To be able to do a continuous Fourier transform on a signal before and after sampling

**Discrete signal**

\[ f_k = f(k \Delta T), \quad k = 0, \pm 1, \ldots \]

**Samples from continuous function**

\[ f_k = f(k \Delta T) \]

**Representation as a function of \( t \)**

- Multiplication of \( f(t) \) with Shah

\[ f(t) = f(t) \delta_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} f_k \delta(t - k \Delta T) \]

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**Scenario**

* Consider a continuous function \( f(t) \) that we have to sample at uniform intervals \( (\Delta T) \) of the independent variable \( t \).

* Assume that \( f(t) \) extends from \(-\infty \) to \( \infty \) w.r.t \( t \).

* To model the sampling process, we need to multiply \( f(t) \) by a **sampling function**

\[ \tilde{f}(t) = f(t) S_{\Delta T}(t) \]

the sampled function

\[ = \sum_{k=\infty}^{\infty} f(t) S(t - k \Delta T) \]

**The sampled function**

- Each component is an impulse weighted by the value of \( f(t) \) at the location of the impulse.

\[ \Rightarrow \text{sifting.} \]

let the weight of the impulse at \( t = k \Delta T \) by \( f_k \); it can be obtained by

\[ f_k = \int_{-\infty}^{\infty} f(t) S(t - k \Delta T) = f(k \Delta T) \]

\[ \therefore \tilde{f}(t) = \sum_{k=\infty}^{\infty} f_k S(t - k \Delta T) \]
Fourier Series of A Shah Functional

\[ s(t) = \sum_{k=-\infty}^{\infty} \delta(t - k\Delta T) \]

\[ S(u) = \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} \delta(u - \frac{k}{\Delta T}) = \sum_{k=-\infty}^{\infty} \delta(\Delta Tu - k) \]

\[ \text{The F.T. of a shah functional is still a shah functional} \]

Based on the convolution theorem and the duality of the Fourier transform we have:

\[ S(t) \xrightarrow{F.T.} \frac{1}{\sqrt{2\pi}} \]
\[ s(t - \tau) \xrightarrow{F.T.} e^{-j2\pi\tau u} \]
\[ s(t - k\Delta T) \xrightarrow{F.T.} e^{-j2\pi(k\Delta T)u} \]

**Derivation:**

- The impulse train is periodic in \( \Delta T \), so it can be expressed as a Fourier series:

\[ S_{\Delta T}(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi\frac{k}{\Delta T}t} \]

where

\[ c_k = \frac{1}{\sqrt{\Delta T}} \int_{-\Delta T/2}^{\Delta T/2} S(t) e^{-j2\pi\frac{k}{\Delta T}t} \, dt \]

\[ S_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} e^{j2\pi\frac{k}{\Delta T}t} \]

Now, we need to get the F.T. of this expression.

**But the impulse train is periodic...**

**Its transform is not straightforward as in the case of a single impulse.**

**From F.T. Symmetry** (the only difference is a sign)

\[ g(t) \rightarrow G(u) \]
\[ g(-u) \leftarrow G^*(t) \]

\[ S(t - \tau) \rightarrow e^{-j2\pi\tau u} \]
\[ S(-u - \tau) \leftarrow e^{-j2\pi\tau(-u)} \]

let \(-\tau = \alpha\)

\[ \text{TF} \{ e^{j2\pi\alpha t} \} = S(-u + \alpha) = S(u - \alpha) \]

**Using this and the linearity property of F.T.:**

\[ \text{TF} \{ e^{j2\pi\frac{k}{\Delta T} t} \} = S(u - \frac{k}{\Delta T}) \]

\[ S(u) = \frac{1}{\Delta T} \sum_{k=-\infty}^{\infty} S(u - \frac{k}{\Delta T}) \Rightarrow S(\Delta Tu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} S(u) e^{-j2\pi\tau u} \, du \]
Fourier Transform of A Discrete Sampling

\[ \tilde{f}(t) = f(t) s(t) \quad \longrightarrow \quad \tilde{F}(u) = F(u) \ast S(u) \]

Recall

\[ S(u) = \frac{1}{\Delta T} \sum_{k=0}^{\infty} S(u - \frac{k}{\Delta T}) \]

The convolution of \( F(u) \) and \( S(u) \) can be obtained directly from the convolution definition:

\[ F(u) = F(u) \ast S(u) \]

\[ = \int F(\tau) S(u - \tau) \, d\tau \]

\[ = \frac{1}{\Delta T} \int F(\tau) \sum_{k=0}^{\infty} S(u - \tau - \frac{k}{\Delta T}) \, d\tau \]

\[ = \frac{1}{\Delta T} \sum_{k=0}^{\infty} \int F(\omega) S(u - \tau - \frac{k}{\Delta T}) \, d\tau \]

The F.T of a discrete function is periodic.

What does this mean?

- The F.T of the sampled function \( \tilde{f}(t) \) is an infinite, periodic sequence of copies of \( F(u) \) - the transform of the original continuous function.

The F.T of a discrete function is periodic.

- The separation of those copies is determined by \( \frac{1}{\Delta T} \).

Note: \( \tilde{F}(u) \) is still by itself continuous in the frequency domain.
Fourier Transform of A Discrete Sampling

\[ \hat{F}(\omega) = F(\omega) \ast S(\omega) \]

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By sampling, we actually changing the original continuous function.

You don't know which signal to recover for the given samples.

Energy from higher freq gets folded back down into lower freq - Aliasing
What if $F(u)$ is Narrower in the Fourier Domain?

- No aliasing!
- How could we recover the original signal?

\[ F(u) \]

- If $F(u)$ did not alias a little bit \( \otimes \) and becomes narrower in the Fourier domain.
  
  Such aliasing won't happen.

- The reconstruction of the original continuous function would be as easy as applying a low-pass filter to get just one copy of $F(u)$ from $\hat{F}(u)$. 
What Comes Out of This Model

- Sampling criterion for complete recovery
- An understanding of the effects of sampling
  - Aliasing and how to avoid it
- Reconstruction of signals from discrete samples

We just can't let any continuous function make a kick so that it will fit in a narrower band, we need to modify our sampling function according to the given function in order to avoid this aliasing effect.
Shannon Sampling Theorem

- **Assuming a signal that is band limited:**
  \[ f(t) \leftrightarrow F(u) \quad |F(u)| = 0 \quad \forall \quad |u| > B \]

- **Given set of samples from that signal**
  \[ f_k = f(k\Delta T) \quad \Delta T \leq \frac{1}{2B} \]

- **Samples can be used to generate the original signal**
  - Samples and continuous signal are equivalent

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**Goal:**

- Establish the conditions under which a continuous function can be recovered uniquely from a set of its samples.

**Assume a band limited signal:**

\[ f(t) \rightarrow F(u) \quad \forall \quad |u| > B \]

We can recover \( f(t) \) from its sampled version if we can isolate a copy of \( F(u) \) from the periodic sequence of copies in \( F(u) \).

\[ \Delta T \leq \frac{1}{2B} \quad \text{or} \quad \Delta T \geq \frac{2B}{2B} \]

\[ \text{Sampling Rate} \]

\[ \text{Sampling Period} \]

- A continuous, band-limited function can be uniquely recovered from a set of its samples, if the samples are acquired at a rate exceeding twice its high frequency content.

In practice, sampling need to exceed the Nyquist rate.
Sampling Theorem

- Quantifies the amount of information in a signal
  - Discrete signal contains limited frequencies
  - Band-limited signals contain no more information than their discrete equivalents
- Reconstruction by cutting away the repeated signals in the Fourier domain
  - Convolution with sinc function in space/time

\[ h(u) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \]
\[ h(t) = 1 \text{sinc} \left( \frac{t}{\Delta T} \right) \]
\[ f(t) = \mathcal{F}^{-1} \{ F(u) \} = \mathcal{F}^{-1} \{ H(u) \hat{F}(u) \} = h(t) \ast \hat{f}(t) \rightarrow \text{Convolution theorem} \]
Reconstruction

- Convolution with sinc function

\[ f(t) = \hat{f}(t) * \mathbf{F}^{-1} \left[ \text{rect} \left( \Delta T u \right) \right] \]

\[ = \left( \sum_k f_k \delta(t - k \Delta T) \right) * \text{sinc} \left( \frac{t}{\Delta T} \right) = \sum_k f_k \text{sinc} \left( \frac{t - k \Delta T}{\Delta T} \right) \]

- This shows that the perfectly reconstructed function is an infinite sum of sinc functions weighted by the sample values.

- It has also an important property that the reconstructed function is identically equal to the sample values at multiple integer increments of \( \Delta T \).

\[ \text{u.e. } f(t) = f_k \text{ where } k = k \Delta T \]

Blow sample points, values of \( f(t) \) are interpolations formed by the sum of the sinc functions.
Sinc Interpolation Issues

- Most functions are not band limited
- Forcing functions to be band-limited can cause artifacts (ringing)
Sinc Interpolation Issues

Ringing - Gibbs phenomenon
Other issues:
Sinc is infinite - must be truncated

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Aliasing

- High frequencies appear as low frequencies when undersampled

\[ \frac{1}{\Delta T} < 2B \]

\[ \Rightarrow \text{ undersampled} \]

Q: What happens if a band limited function is sampled at a rate that is less than twice its highest frequency?!!

\[ \Rightarrow \text{ undersampled.} \]

\[ \Rightarrow \text{ The periods/copies begin to overlap it becomes impossible to isolate a single period of the transform, regardless the filter used.} \]

\[ \textbf{Note:} \text{ if the sampling rate is exactly the Nyquist rate} \]

\[ \Rightarrow \text{ 1 sample/second.} \]

let the sampling taken at:

\[ t = -\ldots, -1, 0, 1, 2, 3, \ldots \]

This results in

\[ \sin(-\pi), \sin(0), \sin(\pi) \]

which are all zero.

\[ \Rightarrow \text{That's why we need to exceed Nyquist Rate} \]

* The large dots: samples taken at a rate less than 1 sample/second.

\[ \Delta T = 2 \text{ sec.} \]

The sampled signal looks like a sine wave but with \( \frac{1}{2} \) the frequency of the original signal.

\[ \Rightarrow \text{No way to know the original } f(t) \text{ given its samples.} \]
Aliasing

In 2D

Sampling. Checkerboards with different sizes of the squares.

Reducing the square to slightly less than 0.5 pixels → the aliased result looks like a normal checkerboard pattern.

Aliasing can create results that may be misleading → it looks like as if the square size was taken on 1/2 pixels.

When the size of the square is reduced to slightly less than one camera pixel → a severely aliased image results.

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Overcoming Aliasing

- Filter data prior to sampling
  - Ideally - band limit the data (conv with sine function)
    \( \Rightarrow \text{Ideal LPF} \)
  - In practice - limit effects with fuzzy/soft low pass
    \( \Rightarrow \text{Non-Ideal LPF} \)

* Effects of aliasing can be reduced by slightly defocusing (blurring) the scene to be digitized.

  \( \Rightarrow \text{Attenuate high frequencies.} \)

* Anti-Aliasing has to be done at the “front-end” before the image is sampled.

* There are no such things as after-the-fact software anti-aliasing filters that can be used to reduce the effects of aliasing (caused by violations of the Sampling theorem).
Antialiasing in Graphics

- Screen resolution produces aliasing on underlying geometry

Multiple high-res samples get averaged to create one screen sample

aliased antialiased

Sampling Aperture → This scene area will be fitted to a pixel → averaged.
Antialiasing

Aliasing on resampled images.

Original (negligible visual aliasing)

Resized to 50% (pixel deletion)

Aliasing is visible.

Blurring the image with a 3x3 average filter then resize.

Image is more blurred, yet aliasing disappears.
Interpolation as Convolution

- Any discrete set of samples can be considered as a functional
  \[ f(t) = \sum_k f_k \delta(t - k\Delta t) \]

- Any linear interpolant can be considered as a convolution
  - Nearest neighbor - rect(t)
  - Linear - \( \text{tri}(t) \)

\( \text{tri}(t) = \begin{cases} 
  t+1 & -1 \leq t \leq 0 \\
  1-t & 0 < t < 1 \\
  0 & \text{otherwise}
\end{cases} \)

1. Nearest Neighbor:
   \[ h(t) = \text{rect}(t) = \begin{cases} 
  1 & |t| < \frac{1}{2} \\
  0 & \text{otherwise}
\end{cases} \]

2. Linear:
   \[ h(t) = \text{tri}(t) = \begin{cases} 
  t+1 & -1 \leq t \leq 0 \\
  1-t & 0 < t < 1 \\
  0 & \text{otherwise}
\end{cases} \]

- Sinc function handled using box or triangle filter (defined in spatial domain) instead of the sinc function as approximation to the infinite interpolation.

- \( F(H) = F(f) \ast h(t) \) can be considered as interpolation.

One way to handle the infinite compact of the sinc function (ideal LPF) is to truncate it with a box (rect) function in the spatial domain \( \Rightarrow \) ringing.
Convolution-Based Interpolation

- Can be studied in terms of Fourier Domain
- Issues
  - Pass energy (=1) in band
  - Low energy out of band
  - Reduce hard cut off (Gibbs, ringing)