Systems Theory

$$g(x) \longrightarrow \boxed{} \longrightarrow h(x)$$

$$\uparrow$$

$$h(x) = T[g(x)]$$

linear
shift invariant

Linear

$$l(x) = T[k(x)]$$
$$h(x) = T[g(x)]$$

$$T(k+g) = l + h$$

linear operations on $g(x)$ is all possible multiplications + additions on values of $g(x)$

$$h(x) = \int_{-\infty}^{\infty} f(\alpha, x)\, g(\alpha)\, d\alpha$$

Shift Invariant

$$h(x) = T[g(x)] \Rightarrow h(x-\tau) = T[g(x-\tau)]$$

$$h(x-\tau) = \int_{-\infty}^{\infty} f(\alpha, x)\, g(\alpha-\tau)\, d\alpha$$

$$\alpha' = \alpha - \tau$$

$$h(x-\tau) = \int f(\alpha'+\tau, x)\, g(\alpha')\, d\alpha'$$

$$x' = x - \tau$$

$$h(x') = \int f(\alpha'+\tau, x'+\tau)\, g(\alpha')\, d\alpha'$$

$\Downarrow$ change variables

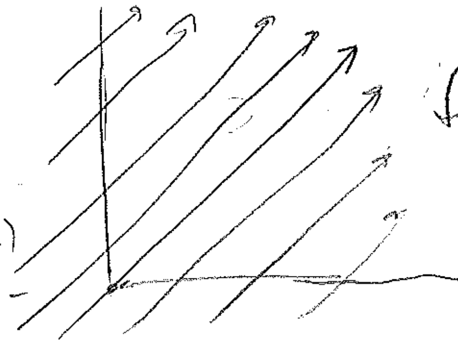
$$h(x) = \int_{-\infty}^{\infty} f(\alpha+\tau, x+\tau)\, g(\alpha)\, d\alpha.$$

must hold for any $g(x)$

$$= \int f(\alpha, x)\, g(\alpha)\, d\alpha.$$

$$\Rightarrow f(\alpha+\tau, x+\tau) = f(\alpha, x) \quad \forall\, \alpha, x$$

$f(x,\alpha)$



f is constant along lines $x-\alpha = k$

$$f(x,\alpha) = f(x-\alpha)$$

$$h(x) = \int_{-\infty}^{\infty} f(x-\alpha)\, g(x)\, dx$$

Thus all S.I., Lin systems are convolutions defined by "characteristic function" or "kernel".

Fourier Transform

Review complex numbers.

Tool for analysis, algorit
processing

A language for describing functions

Continuous fourier transform

$$\mathbb{F}\{f(t)\} = F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-j2\pi st}\, dt$$

$$\mathbb{F}^{-1}\{F(s)\} = \int_{-\infty}^{\infty} F(s)\, e^{j2\pi st}\, dt$$

Fourier's integral theorem:

$$f(t) = \mathbb{F}^{-1}\{\mathbb{F}\{f(t)\}\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t)\, e^{-j2\pi st}\, dt\right] e^{j2\pi st}\, ds$$

Beware: Alternative ways of writing these integral eqs. depending on where you put the 2π (different constants in front).

- our convention — s is measured in cycles. (rather than radians).

Ex: Fourier trans of a Gaussian

$$f(t) = e^{-\pi t^2}$$

$$\mathbb{F}(s) = \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j 2\pi s t}\, dt = \int_{-\infty}^{\infty} e^{-\pi(t^2 + j2st)}\, dt$$

complete the square

$$= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t + js)^2}\, dt$$

$$t + js \rightarrow u \qquad dt \rightarrow du$$

$$\mathbb{F}(s) = e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi u^2}\, du \longleftarrow \text{unity (from probability)}$$

$$= e^{-\pi s^2}$$

Properties of FT.

Evenness & Oddness

Even $\qquad f_e(t) = f_e(-t)$.

Odd

$$f_o(t) = -f_o(-t)$$

Every function has odd & even components.

$$f_e(t) = \frac{1}{2}\left[f(t) + f(-t)\right]$$

$$f_o(t) = \frac{1}{2}\left[f(t) - f(-t)\right]$$

$$f(t) = f_e(t) + f_o(t)$$

Euler relation

$$e^{jx} = \cos(x) + j\sin(x)$$

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-j2\pi st}\,dt = \int_{-\infty}^{\infty} f(t)\cos(2\pi st)\,dt$$

$$-j\int_{-\infty}^{\infty} f(t)\sin(2\pi st)\,dt$$

odd/even

$$F(s) = \int_{-\infty}^{\infty} f_e(t)\cos(2\pi st)\,dt + \int_{-\infty}^{\infty} f_o(t)\cos(2\pi st)\,dt$$

$$-j\int_{-\infty}^{\infty} f_e(t)\sin(2\pi st)\,dt - j\int_{-\infty}^{\infty} f_o(t)\sin(2\pi st)\,dt$$

odd/even

$$F(s) = \int_{-\infty}^{\infty} f_e(t)\cos(2\pi st)\,dt - j\int_{-\infty}^{\infty} f_o(t)\sin(2\pi st)\,dt$$

$$= F_e(s) + j F_o(s)$$

	$\mathcal{F}$	
odd	$\longleftrightarrow$	odd
even	$\longleftrightarrow$	even
odd	$\longleftrightarrow$	"j"

Conjugate symmetry

$$F(s) = F^*(-s)$$

$$\mathbb{F}^{-1}\{F(s)G(s)\} = f(t) \otimes g(t).$$

Fourier transform of impulse

$$f(t) \otimes \delta(t) = f(t)$$

$$\Downarrow$$

$$F(s)\,\mathbb{F}\{\delta(t)\} = F(s)$$

$$\Rightarrow \mathbb{F}\{\delta(t)\} = 1$$

Similarity Theorem

Change of scale of abscissa

$$\mathbb{F}\{f(at)\} = \int_{-\infty}^{\infty} f(at)\, e^{-j2\pi st}\, dt$$

$$= \frac{1}{|a|}\, F\left\{\frac{s}{a}\right\}.$$

time longer vs. smaller.

Addition

$$\mathbb{F}\{f(t)+g(t)\} = \int_{-\infty}^{\infty} [f(t)+g(t)]\, e^{-j2\pi st}\, dt$$

$$= F(s) + G(s).$$

$$\mathbb{F}\{c\, f(t)\} = c\, F(s)$$

Shift

$$\mathbb{F}\{f(t-a)\} = e^{-j2\pi as} F(s).$$

position information is contained in phase in FT.

Convolution Theorem

$$\mathbb{F}\{f(t) \otimes g(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u)\, g(t-u)\, du\, e^{-j2\pi st}\, dt$$

$$= \int_{-\infty}^{\infty} f(u) \int_{-\infty}^{\infty} g(t-u)\, e^{-j2\pi st}\, dt\, du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-j2\pi su}\, G(s) = G(s) \int_{-\infty}^{\infty} f(u)\, e^{-j2\pi su}\, du$$

$$= F(s)\, G(s).$$

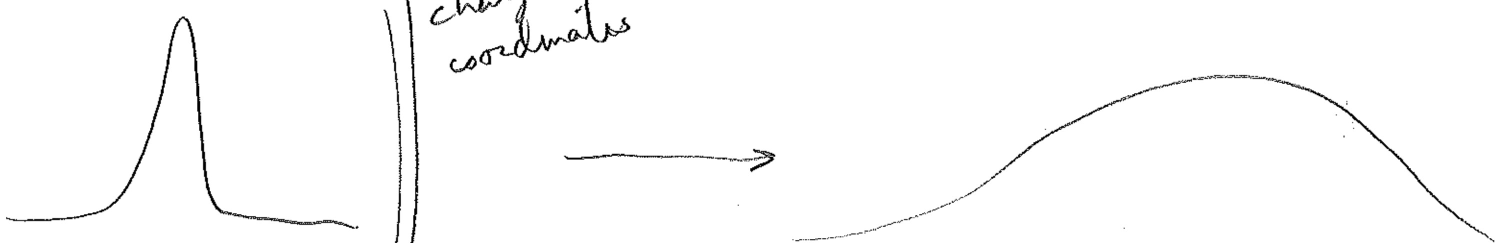
e.g. Gaussian

$$\mathcal{F}\{e^{-\pi t^2}\} = e^{-\pi s^2}$$

$$\Downarrow \text{ sim. theorem}$$

$$\mathcal{F}\{e^{-\pi (at)^2}\} = \frac{1}{a} e^{-\pi (s/a)^2}$$

change coordinates



$$\mathcal{F}\{e^{-t^2/2\sigma^2}\} = \sqrt{2\pi\sigma^2}\, e^{-s^2/2\alpha^2}$$

$$\alpha = \frac{1}{2\pi\sigma}$$

$$\begin{array}{c} f(t) \\ F(s) \end{array} \longrightarrow \boxed{\begin{array}{c} g(t) \\ G(s) \end{array}} \longrightarrow \begin{array}{c} h(t) \\ H(s) \end{array}$$

$$H(s) = F(s)G(s)$$

$$G(s) = \frac{H(s)}{F(s)}$$

$$F(s) \neq 0$$

$$g(t) = \mathcal{F}^{-1}\left\{\frac{H(s)}{F(s)}\right\}$$

Existence of the FT

what kinds of functions will this work on?

Transient functions (short lived).

$$\int_{-\infty}^{\infty} |f(t)|\,dt < \infty \quad \longleftrightarrow L_1$$

& finite # of discontinuities.

Periodic functions

inverse transform of impulse pair

$$\mathbb{F}^{-1}\{\delta(s-f_0)+\delta(s+f_0)\} = \int_{-\infty}^{\infty} [\delta(s-f_0)+\delta(s+f_0)]e^{j2\pi st}\,ds.$$

$$f(t) = e^{-j2\pi f_0 t} + e^{+j2\pi f_0 t} = 2\cos(2\pi f_0 t).$$

$\longrightarrow$

$$\cos(2\pi wt) \longleftrightarrow \frac{1}{2}[\delta(s-w)+\delta(s+w)]$$

$$\sin(2\pi wt) \longleftrightarrow \frac{j}{2}[\delta(s+w)-\delta(s-w)]$$

constant

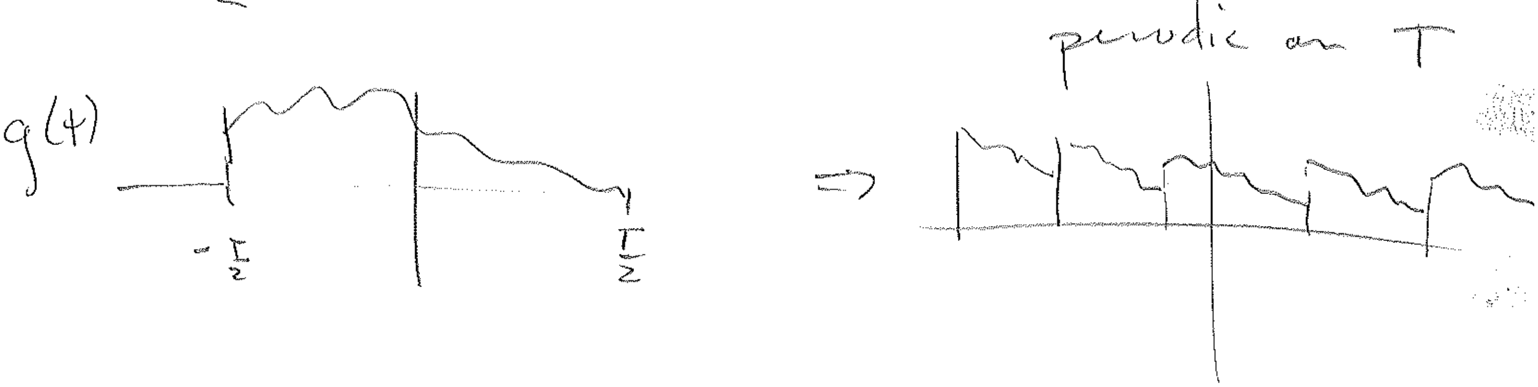
$$\mathbb{F}\{1\} = \delta(s).$$

<u>Thm:</u>
Any periodic function of frequency ω can be expressed as sum of sinusoids of frequency $n\omega$, where $n \in \mathbb{Z}$. FT is linear, therefore any periodic function has FT consisting of a train of equally spaced impulses.

<u>Fourier series</u>

consider a function with compact support that is zero outside the interval

$[-T/2, T/2]$



Then the Fourier transform is a set of discrete pulses, and we can let s be a discrete variable with

$\Delta s = \frac{1}{T}$

$$G_n = G(n\Delta s) = \int_{-T/2}^{T/2} g(t) e^{-j2\pi(n\Delta s t)} dt$$

infinitely-many, complex-valued coefficients

inverse is

$$g(t) = \sum_{n=0}^{\infty} G(n\Delta s) e^{j2\pi(n\Delta s t)} \Delta s = \frac{1}{T}\sum_{n=0}^{\infty} G_n e^{j2\pi\left(\frac{n}{T}t\right)}$$

Thus, g(t) is a sum of sinusoids

The Fourier series expansion of a function f(t) is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(2\pi \frac{n}{T} t\right) + \sum_{n=1}^{\infty} b_n \sin\left(2\pi \frac{n}{T} t\right).$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\cos\left(2\pi \frac{n}{T}x\right)dx \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(x)\sin\left(2\pi \frac{n}{T}x\right)dx$$

Discrete Fourier Transform

Discrete time & frequency.

$$G_n = G(n\Delta s) = \sum_{i=-N/2}^{N/2} g(i\Delta t)e^{-j2\pi(n\Delta s)i\Delta t}\Delta t$$

$$= \frac{T}{N}\sum_{i=-N/2}^{N/2} g_i e^{-j2\pi\left(\frac{n}{N}\right)i} \qquad T = N\Delta t$$

$$\Delta s = \frac{1}{\Delta t}.$$

$$g_i = g(i\Delta t) = \sum_{n=-\infty}^{\infty} G(n\Delta s)e^{j2\pi(n\Delta s)i\Delta t}\Delta s$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} G_n e^{j2\pi\left(\frac{i}{N}\right)n}$$

If we simply have sequence of length N

let $T = ~~rN~~$

$$F_n = \frac{1}{N^{\frac{1}{2}}}\sum_{i=0}^{N-1} f_i \cdot e^{-j2\pi\frac{n}{N}i} \qquad f_i = \frac{1}{N^{\frac{1}{2}}}\sum_{n=0}^{N-1} F_n e^{j2\pi\frac{i}{N}n}$$

The CFT & DFT are essentially equivalent

Fast Fourier Transform

Implementation of DFT is $O(N^2)$

Can be thought of as matrix

$\bar{F} = W \breve{f}$

matrix has redundancy.
Product of sparse matrices

$O(N \log N)$.

Matrix Formulation

DFT in dir 2.

DFT in dir 1

$$G = F^{-1} g F$$

$$F = [f_{ik}] = \left[\frac{1}{\sqrt{N}} e^{-j2\pi ik/N}\right]$$

N × N kernel matrix

$$g = [g(i,k)]$$

matrix – no necessary to do row stacking

The F.T. in 2D.

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\, dy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)}\, du\, dv$$

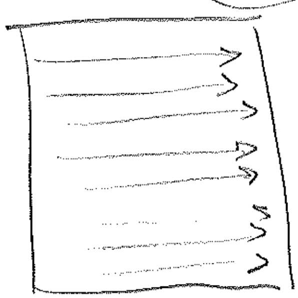
D.FT. in 2D.

$$G(m,n) = \frac{1}{N}\sum_{i=0}^{N-1}\sum_{k=0}^{N-1} g(i,k)\, e^{-2\pi\left(m\frac{i}{N} + n\frac{k}{N}\right)}.$$

$$g(i,k) = \frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} G(m,n)\, e^{j2\pi\left(i\frac{m}{N} + k\frac{n}{N}\right)}.$$

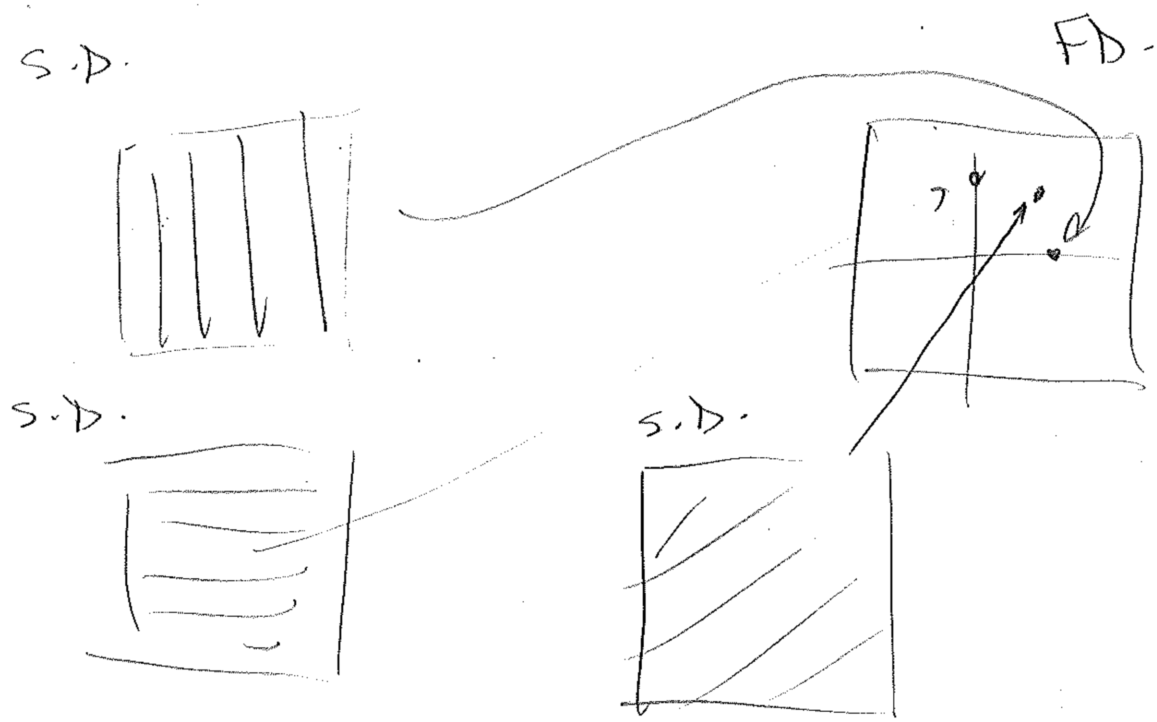
Separability

$$\underbrace{G(m,n) = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}\Big[\underbrace{\frac{1}{\sqrt{N}}\sum_{k=0} g(i,k)\, e^{-j2\pi\left(n\frac{k}{N}\right)}}_{G(i,n)}\Big] e^{-j2\pi\left(m\frac{i}{N}\right)}}_{G(m,n)}$$



In 2D F.T., the frequency is really a vector $\bar{\omega} = \begin{bmatrix} u \\ v \end{bmatrix}$.

$\bar{\omega}$ corresponds to a harmonic with both a frequency and 2D orientation



Prop	S.D.	F.D.
Addition		
Similarity	$f(ax, by)$	$\frac{1}{\lvert ab \rvert} F\left(\frac{u}{a}, \frac{v}{b}\right)$
Shift	$f(x-a, y-b)$	$e^{-j2\pi(au+bv)} F(u,v)$
Convolution Thm	$f(x,y) \circledast g(x,y)$	$F(u,v)\,G(u,v)$
Separable product	$f(x)\,g(y)$	$F(u)\,G(v)$
Differentiation	$\left(\frac{\partial}{\partial x}\right)^n \left(\frac{\partial}{\partial y}\right)^m f(x,y)$	$(j2\pi u)^n (j2\pi v)^m F(u,v)$
Rotation	$f\left(R\begin{bmatrix} x \\ y \end{bmatrix}\right)$	$F\left(R\begin{bmatrix} u \\ v \end{bmatrix}\right)$

Differentiation (for transient signals).

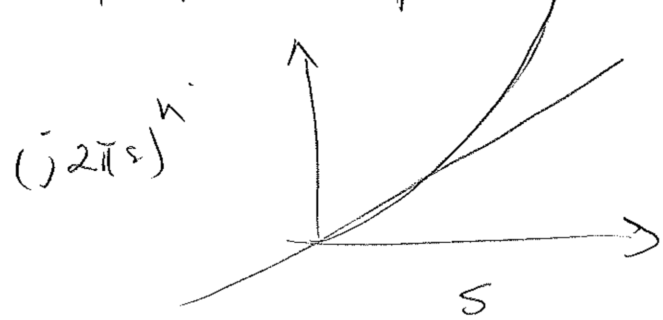
$$\mathcal{F}\{f'(t)\} = \int_{-\infty}^{\infty} \underbrace{\frac{d}{dt} f(t)}_{dv} \, \underbrace{e^{-j2\pi st}}_{u} \, dt$$

$$= f(t) e^{-j2\pi st} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(t) \frac{d}{dt} e^{-j2\pi st} \, dt =$$

$$(j2\pi s)\int_{-\infty}^{\infty} f(t)\, e^{-j2\pi st}\, dt = (j2\pi s)\, F(s)$$

$$\mathbb{F}\left\{ \frac{d^n}{dt^n} f(t) \right\} = (j2\pi s)^n F(s).$$

- Differentiation corresponds to multiplication by polynomials in frequency domain.
- Thus, differentiation looks like convolution with certain kinds of "funny" operators.
  - derivatives of impulse fctn.
- Differentiation does high frequency enhancement

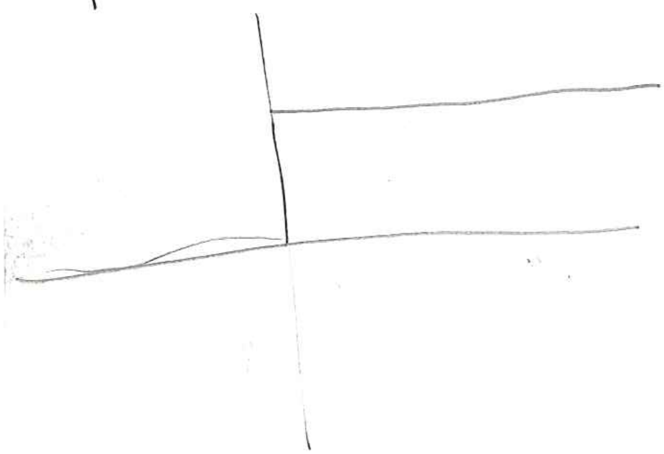


Integration Theorem

$$g(t) = \int_{-\infty}^{t} f(p)\,dp$$

$$\mathcal{F}[g(t)] = \mathcal{F}\left[\int_{-\infty}^{t} f(p)\,dp\right] = \frac{F(\omega)}{j\omega} + \pi F(0)\,\delta(\omega)$$

Step function



← integral of delta function

$$\frac{1}{j\omega} + \pi\,\delta(\omega)$$

Projection

"Collapse" a function by integration

$$p(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$

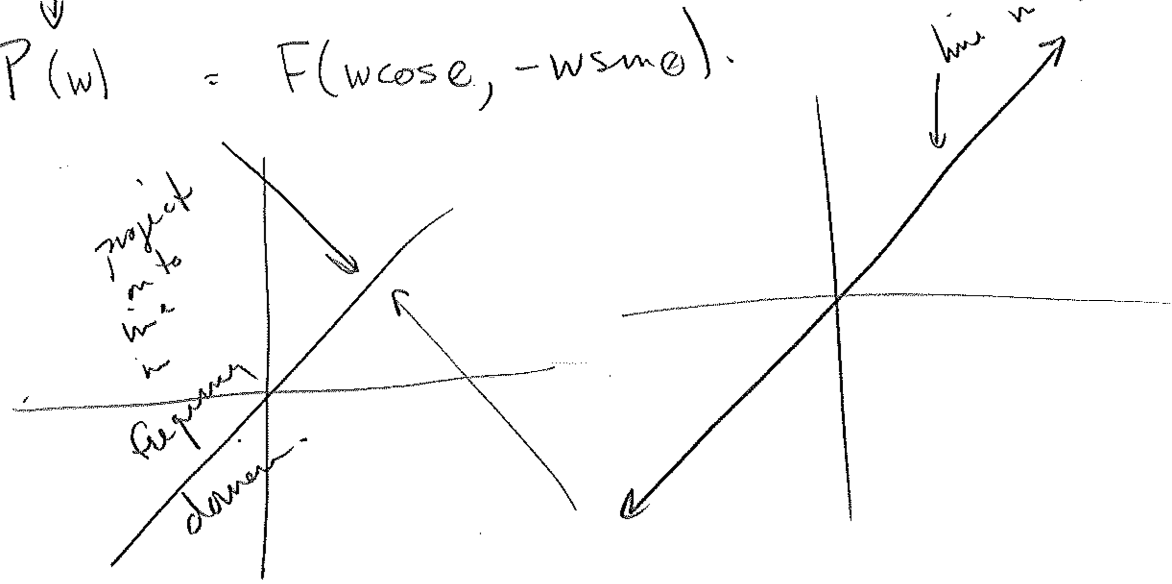
$$P(u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\, e^{-j2\pi xu}\,dx = \iint f(x,y)\,dy\, e^{-j2\pi xu} e^{j2\pi y0}\,dx$$

$$= F(u,0)$$

$$p(z) \qquad z = x\cos\theta + y\sin\theta$$

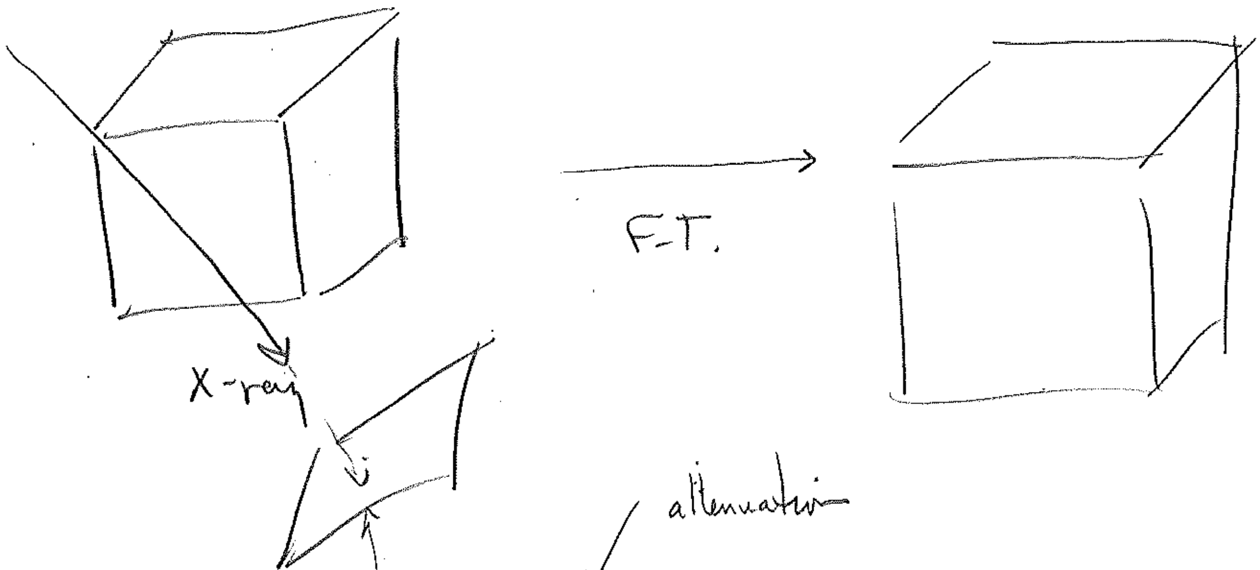
$$\updownarrow$$

$$P(w) = F(w\cos\theta, -w\sin\theta).$$



Application: Synthetic reprojection

CT Volume - set of densities



attenuation

$$\int A(x,y,z)\,d\vec{s} \longrightarrow \text{Image}$$

"Fast" synthetic reprojection by taking "slices" of F.T. (through origin).

Auto correlation:

$$f(t)\otimes f(t) = \int_{-\infty}^{\infty} f(\tau)\, f(t-\tau)\, d\tau$$ self convolution

how do we know?

$$R_f(t) = f(t) \otimes f(-t) = \overbrace{\int_{-\infty}^{\infty} f(\tau)\, f(-t+\tau)\, d\tau}^{\text{even}}$$

$$= \int_{-\infty}^{\infty} f(\tau)\, f(t+\tau)\, d\tau$$

$$\int_{-\infty}^{\infty} R_f(\tau)\,d\tau = \left[\int_{-\infty}^{\infty} f(t)\,dt\right]^2$$

$R_f(t)$ has maximum at $t=0$.

Power Spectrum

$$P_f(s) = \mathbb{F}\{R_f(\tau)\} = \mathbb{F}\{f(t) \otimes f(-t)\}$$

$$= F(s)F(-s) = F(s)F^*(s) = |F(s)|^2$$

"power spectral density"

Cross Correlation

$$R_{fg}(\tau) = f(t) \otimes g(t) = \int_{-\infty}^{\infty} f(t)\,g(t+\tau)\,dt$$

$$\mathbb{F}_{fg}(s) = F(s)\,G^*(s).$$

Frequency Domain

- High frequency cutoff
  - I.e. "Box"
  - Causes
- Gaussian

  Gaussian ←

  $\sigma \longleftrightarrow$

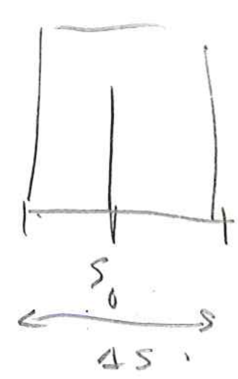
See ppt Slides

Bandpass & Bandstop filters

"Ideal" Bandpass filter

[Signal/noise may occur in different ranges of frequencies

$$G(s) = \begin{cases} 1 & f_1 \le |s| \le f_2 \\ 0 & \text{otherwise} \end{cases}$$



$$G(s) = \Pi\left(\frac{s}{\Delta S}\right) \otimes \left(\delta(s-s_0) + \delta(s+s_0)\right)$$

$$g(t) = \Delta S \, \frac{\sin(\pi \Delta S t)}{\pi \Delta S t} \, 2\cos(2\pi s_0 t)$$

Band-stop filter

$$G(s) = 1 - \Pi\left(\frac{s}{\Delta s}\right) * \left[\delta(s - s_0) + \delta(s + s_0)\right].$$

$$g(t) = \delta(t) - 2\Delta s \, \frac{\sin(\pi \Delta s t)}{\pi \Delta s t} \cos(2\pi s_0 t).$$

General band-pass filter

$$G(s) = K(s) * [\delta(s - s_0) + \delta(s + s_0)].$$

$$\Updownarrow$$

$$g(t) = 2K(t)\cos(2\pi s_0 t)$$

underlying / envelope function

- box
- gaussian
- Anything

Filter Design

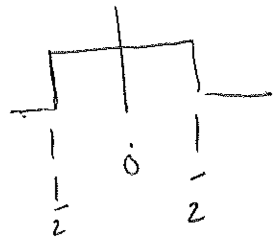
Low-pass.

local averaging in spatial domain

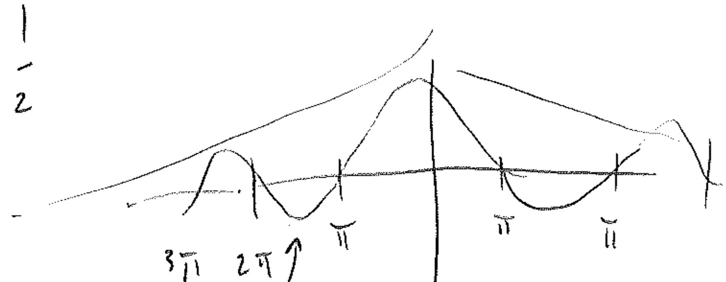
ex:

S.D.

1) Box (unit) $\Pi(x) =$

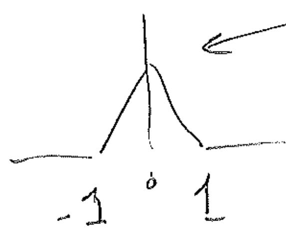


$\longleftrightarrow \left(\frac{\sin s}{s}\right)$



can get "image reversal"

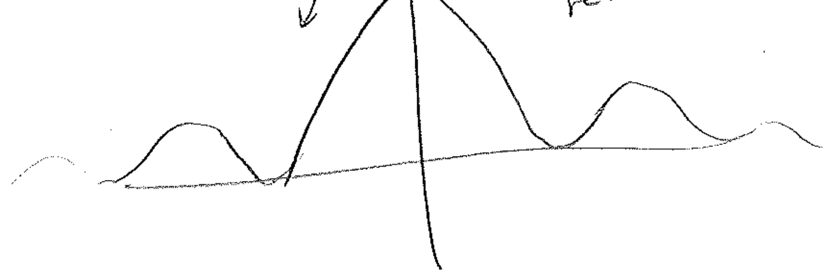
2) Triangle



$\longleftrightarrow \left(\frac{\sin s}{s}\right)^2$

dies out more quickly

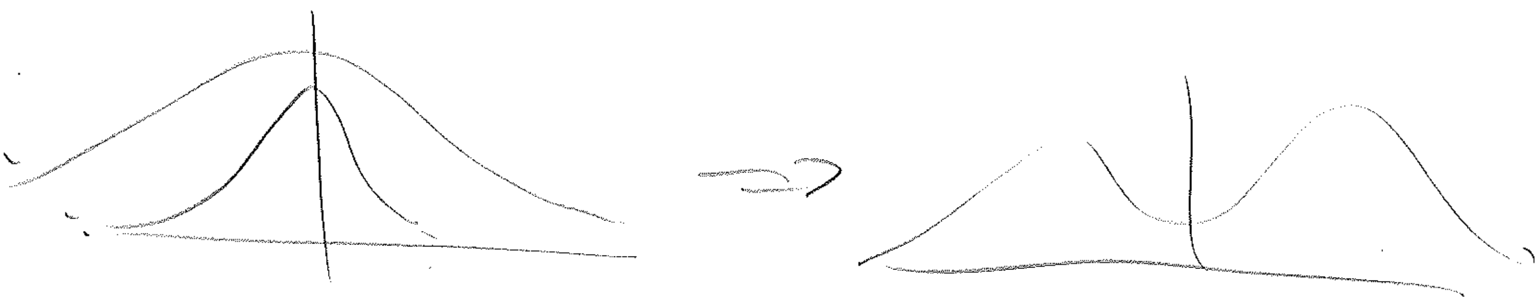
NO image reversal.



High-pass filters

Ex:

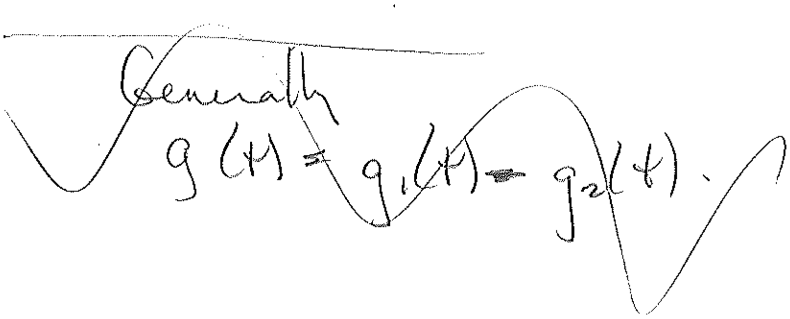
DOG's (Diff. of Gaussians).



$$G(s) = A e^{-s^2/2\sigma_1^2} - B e^{-s^2/2\sigma_2^2}$$

$G(t) =$

"smooth"
"no ringing"
not very selective



Generally
$$g(t) = g_1(t) - g_2(t).$$

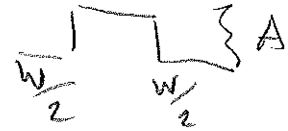
Fourier series in Complex Domain

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi}{T}t}$$

$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j\frac{2\pi}{T}t}\,dt \qquad n = 0, \pm 1, \pm 2, \ldots$$

Box function

$$F(s) = \int_{-\infty}^{\infty} f(t)e^{-j2\pi st}\,dt$$



$$= \int_{-\frac{W}{2}}^{\frac{W}{2}} A e^{-j2\pi st}\,dt = \frac{-A}{j2\pi s}\left[e^{-j2\pi st}\right]_{-\frac{W}{2}}^{\frac{W}{2}}$$

$$= -\frac{A}{j2\pi s}\left[e^{-j\pi sw} - e^{+j\pi sw}\right]$$

$$= \frac{A}{j2\pi s}\left[e^{j\pi sw} - e^{-j2\pi sw}\right] = AW\frac{\sin(\pi sW)}{\pi sW}$$

$W = 1$, $A = 1$ unit box

$$\Rightarrow F(s) = \frac{\sin \pi s}{\pi s} = \operatorname{sinc}(s).$$

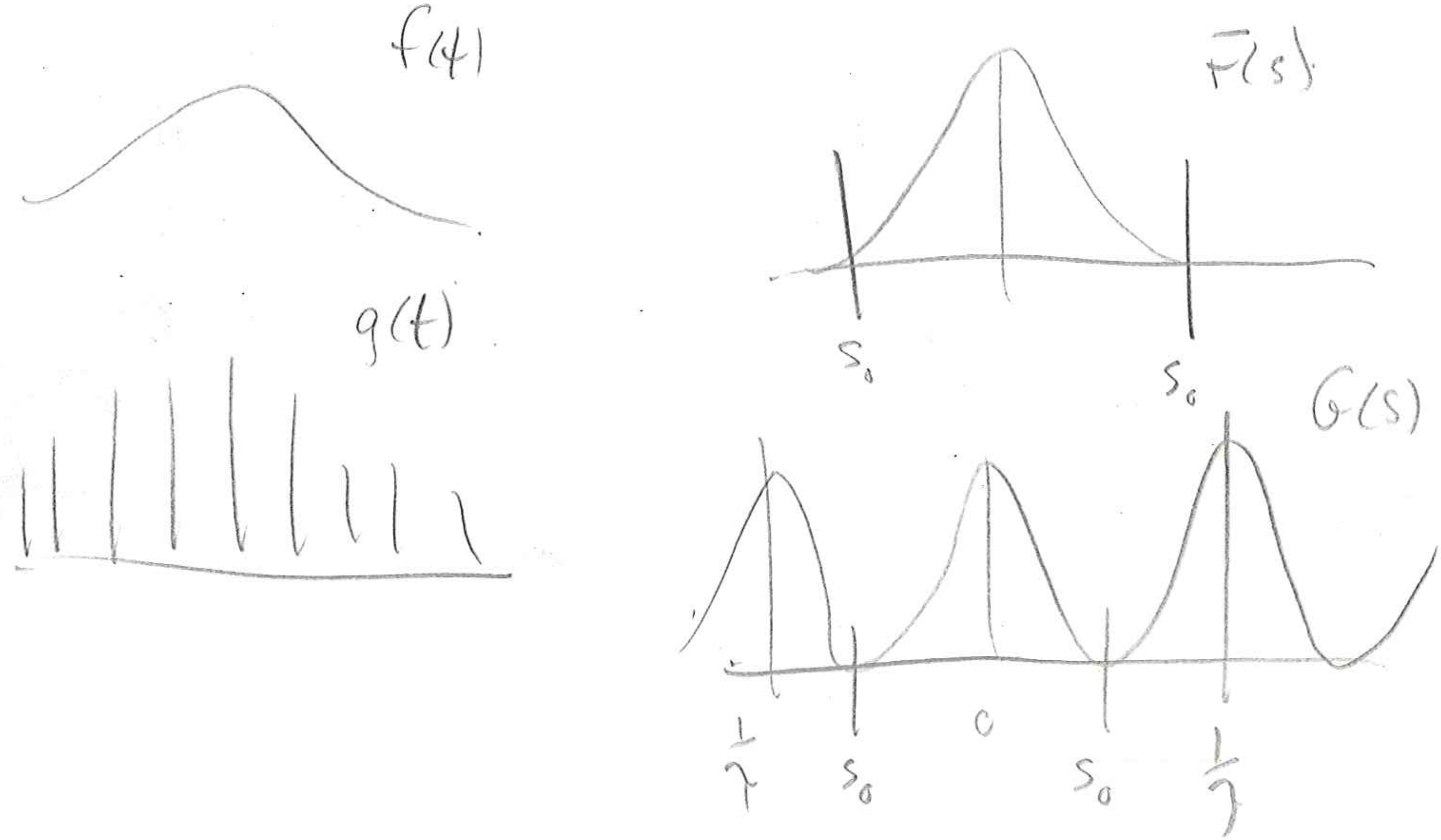
Band limited signals

$$F(s) = 0 \quad |s| \geq s_0$$

See ppt Slides!

Discrete sampling is like

$$f(x)\,\text{III}\left(\frac{x}{\tau}\right) \Rightarrow F(s) \otimes \tau\,\text{III}(\tau s)$$



Sampling Theorem

choose S_1 s.t. $S_0 \le S_1 \le \frac{1}{\tau} - S_0$

Then

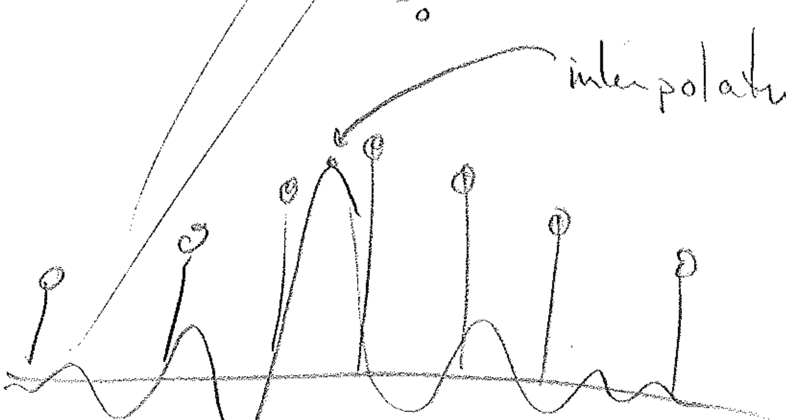
$$F(s) = G(s)\,\Pi(s/2S_1) \quad \& \quad f(t) = \mathcal{F}^{-1}\{F(s)\}$$

and a discretely sampled signal can be completely recovered.

$$f(x) = g(x) \otimes 2S_1 \frac{\sin(2\pi S_1 x)}{2\pi S_1 x}$$

The condition on sampling rate is

$$\tau \le \frac{1}{2S_0}$$



Other page.

Discrete sampling

1) Sampling $\longrightarrow$ loss of information
2) Discrete $\longrightarrow$ continuous
3) How finely to sample.
4) Sampling $\longrightarrow$ spectrum
5) Errors, approximations, etc.

Shah function: $\text{Ш}(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$.

$\mathcal{F}\{\text{Ш}(x)\} = \text{Ш}(s)$.

similarity

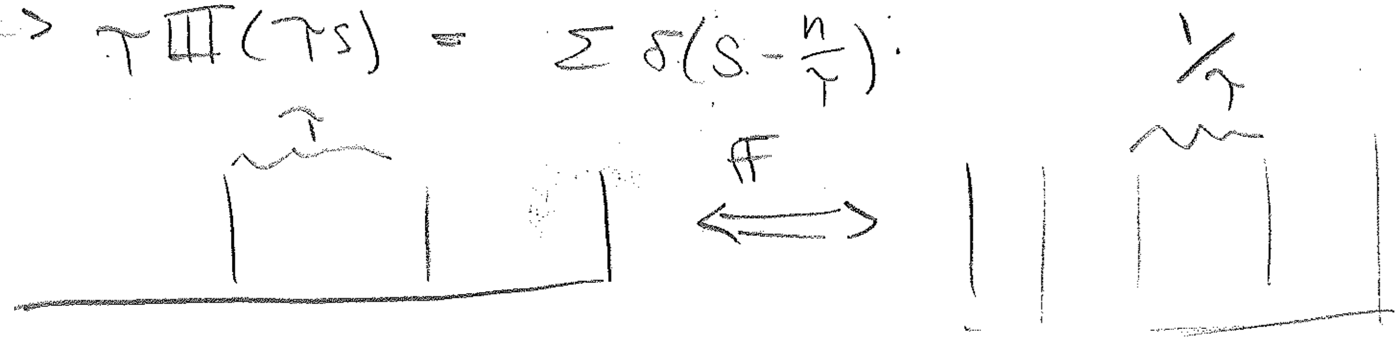
$\mathcal{F}\{\text{Ш}(\frac{x}{T})\} = T\,\text{Ш}(Ts)$

note:

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\Rightarrow \quad \text{Ш}(as) = \frac{1}{|a|} \sum \delta\left(s - \frac{n}{a}\right).$$

$$\Rightarrow \quad T\,\text{Ш}(Ts) = \sum \delta\left(s - \frac{n}{T}\right).$$



Sampling theorem.

cycles/time

If a function $f(t)$ contains no frequencies higher than S, then it is completely determined by any set of regularly spaced samples that are $\Delta t \le \frac{1}{2S}$ apart.

Reconstruction:

from a set of samples

$f_i \quad [-\infty, \ldots -1, 0, 1, \ldots \infty]$

$$f(t) = \sum f_i \delta(t - i\Delta t) \otimes \frac{\sin 2\pi S t}{2\pi S t}$$

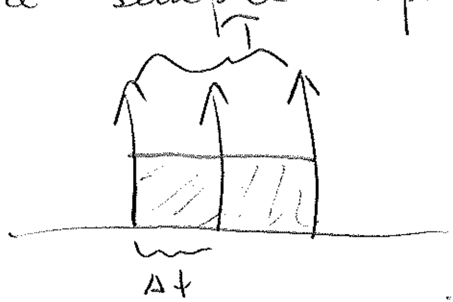
$\frac{1}{2S}$ - is called Nyquist rate.

Controlling Aliasing Error.

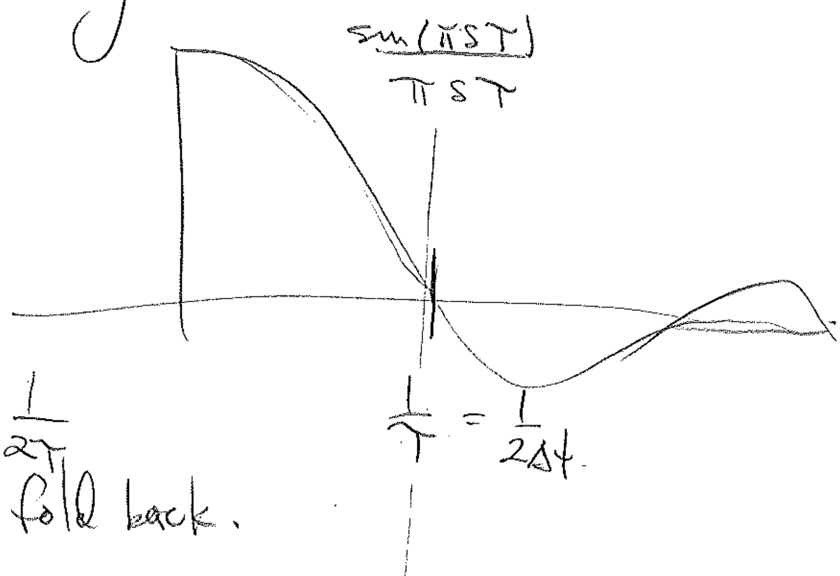
Sample spacing
Sampling aperature (prefiltering).

Antialiasing filter

- Rectangular sampling aperature of width twice sample spacing



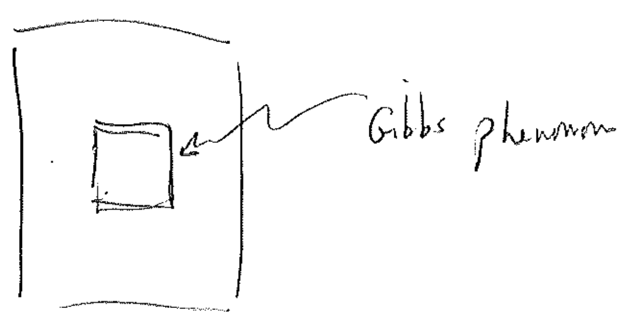
First zero crossing of transfer function at $\frac{1}{2\Delta t}$



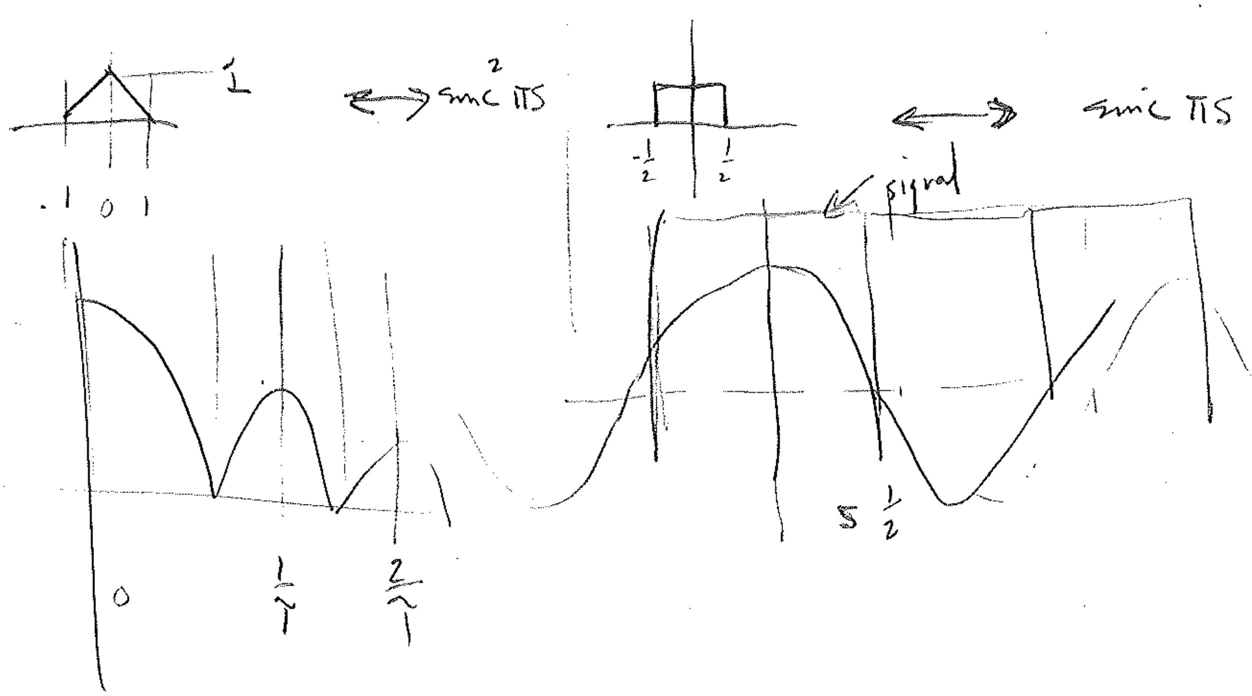
Frequencies above $\frac{1}{2T}$ limited → less fold back.

Spatial domain

- $\mathrm{sinc}(\pi x)$ has infinite extent.
- must truncate — $\mathrm{rect}\left(\frac{x}{2}\right)\mathrm{sinc}(\pi x)$
- ringing — not visually ideal

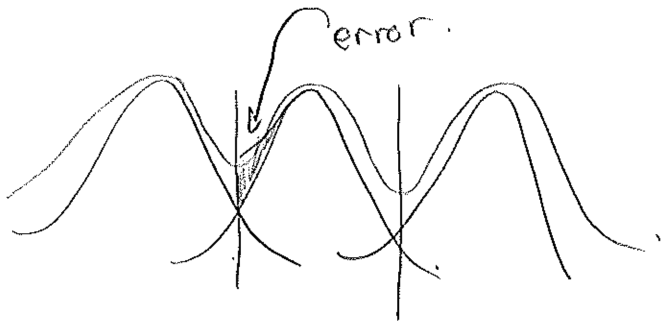


Linear Interpolation | Nearest Neighbor



What if $T \geq \frac{1}{2S_0}$? Undersampling

$$G(s)\,\Pi\left(\frac{s}{2S_1}\right) \neq F(s)$$



Aliasing artifacts

High frequency features associated with folding back of energy for from frequencies

$$s \leq \frac{1}{2T}$$

- "Mock features" – oscillations not in original
- jaggies at edges/high-contrast regions

Truncation

Time domain

Total interval of discrete signal

$T = N\Delta t$ T - truncation window

Same as setting $f(t) = 0$ outside window.

Can do DFT of $f(t)$ samples.

$F(s)$ periodic with period $1/\Delta t$

$$-\frac{1}{2\Delta t} \le s \le \frac{1}{2\Delta t} \qquad \Delta s = \frac{1}{T} = \frac{1}{N\Delta t}$$

$$\Rightarrow N\Delta s = \frac{1}{\Delta t} \quad \text{agreement.}$$

Aliasing

Unavoidable:

Signals of finite extent are not band limited

Truncation in space/time foils attempts at reconstruction

Band Error

Interpolation - (explain)

Sampling theorem suggests an ideal interpolation kernel.

$$s_1 = \frac{1}{2T}$$

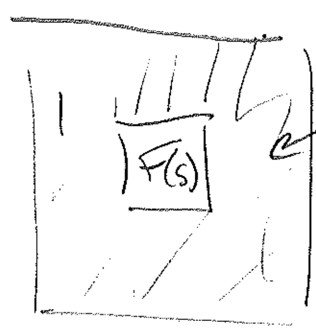
$$\frac{1}{T} \frac{\sin\left(\pi \frac{x}{T}\right)}{\pi \frac{x}{T}} \qquad T=1 \qquad \frac{\sin \pi x}{\pi x} = \mathrm{sinc}(\pi x)$$

can get in between samples by convolving this

Implementations:

Fourier domain

FT padding



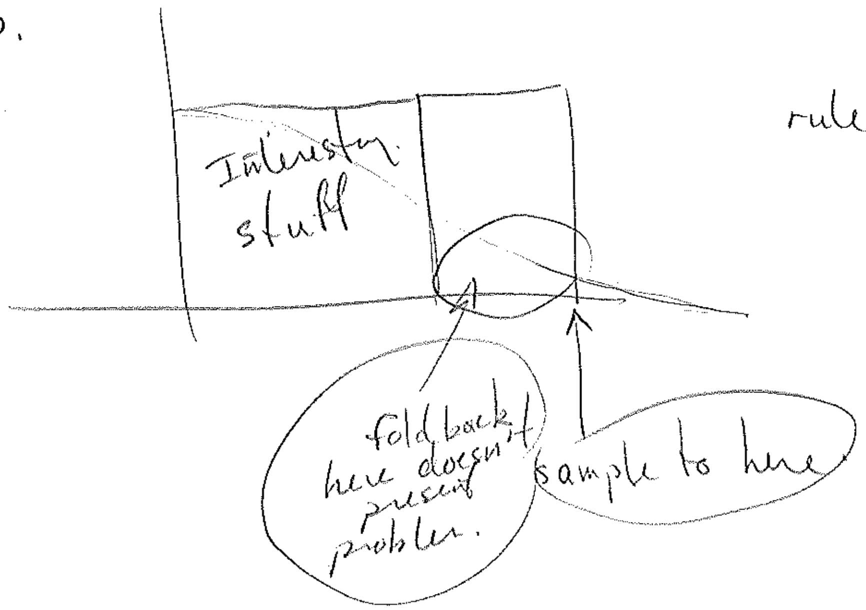
Pad with zeros.

Artifacts?

Oversampling:

Sample at a frequency greater than what you need.

F.D.



rule of thumb: sample at twice relevant frequencies

Typical discrete system