Fourier Series

• J. B. Joseph Fourier, 1807
  – Any periodic function can be expressed as a weighted sum of sines and/or cosines of different frequencies.

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{T} t} \]
Fourier Series

\[ f(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{T} t} \]

\[ c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-j\frac{2\pi n}{T} t} dt \]

- \( f(t) \) periodic signal with period \( T \)
- Frequency of sines and cosines

The complex exponentials form an orthogonal basis for the range \([-T/2,T/2]\) or any other interval with length \( T \) such as \([0,T]\)
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Fourier Transform Pair

\[ F(\mu) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j2\pi\mu t} \, dt \]

\[ f(t) = \mathcal{F}^{-1}\{F(\mu)\} = \int_{\mu=-\infty}^{\mu=\infty} F(\mu) e^{j2\pi\mu t} \, d\mu \]

• The domain of the Fourier transform is the frequency domain.
  – If \( t \) is in seconds, \( \mu \) is in Hertz \((1/\text{seconds})\)

• The function \( f(t) \) can be recovered from its Fourier transform.
Fourier Transform example

- Fourier transform of the box function is the sinc function.
- In general, the Fourier transform is a complex quantity.
- The magnitude of the Fourier transform is a real quantity, called the Fourier spectrum (or frequency spectrum).
Unit impulse function

\[ \delta(t) = \begin{cases} 
\infty & \text{if } t = 0 \\
0 & \text{if } t \neq 0 
\end{cases} \]

- Properties
  - Unit area \( \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \)
  - Sifting \( \int_{-\infty}^{\infty} f(t) \delta(t) \, dt = f(0) \)
  \( \int_{-\infty}^{\infty} f(t) \delta(t - t_o) \, dt = f(t_o) \)
Unit discrete impulse

- **x**: Discrete variable

\[ \delta(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0 
\end{cases} \]

- **Properties**

\[ \sum_{x=-\infty}^{x=\infty} \delta(x) = 1 \]

\[ \sum_{x=-\infty}^{x=\infty} f(x) \delta(x - x_0) = f(x_0) \]
Fourier Transform of Impulses

\[ \mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi\mu t} dt \]

\[ = e^{-j2\pi\mu\nu} = e^0 = 1 \]

\[ \mathcal{F}\{\delta(t - t_o)\} = \int_{-\infty}^{\infty} \delta(t - t_o)e^{-j2\pi\mu t} dt \]

\[ = e^{-j2\pi\mu t_o} \]

\[ = \cos 2\pi\mu t_o - j \sin 2\pi\mu t_o \]
Impulse train

\[ s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} \delta(t - n\Delta T) \]

\[ s_{\Delta T}(t) = \sum_{n=-\infty}^{\infty} c_n e^{j \frac{2\pi n}{\Delta T} t} \]

\[ c_n = \frac{1}{\Delta T} \int_{-\Delta T/2}^{\Delta T/2} \delta(t)e^{-j \frac{2\pi n}{\Delta T} t} dt \]

= \frac{1}{\Delta T} e^{0} = \frac{1}{\Delta T}

- Periodic function so can be represented as a Fourier sum
Fourier Trans. of Impulse Train

\[ s_{\Delta T}(t) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t} \]

\[ \mathcal{F} \{ s_{\Delta T}(t) \} = \mathcal{F} \left\{ \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} e^{j \frac{2\pi n}{\Delta T} t} \right\} \]

\[ = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \mathcal{F} \left\{ e^{j \frac{2\pi n}{\Delta T} t} \right\} \]

\[ = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \delta \left( \mu - \frac{n}{\Delta T} \right) \]

Duality: if \( \mathcal{F}\{f(t)\} \rightarrow F(\mu) \)

then \( \mathcal{F}\{F(t)\} \rightarrow f(-t) \)
Proof of duality for impulses

\[ \mathcal{F}\{\delta(t - t_o)\} = e^{-j2\pi \mu t_o} \quad \text{From before} \]

\[
\mathcal{F}^{-1}\{\delta(\mu - a)\} = \int_{-\infty}^{\infty} \delta(\mu - a)e^{j2\pi \mu t} d\mu
\]

\[
= \int_{-\infty}^{\infty} \delta(-\mu + a)e^{j2\pi \mu t} d\mu
\]

\[
= \int_{-\infty}^{\infty} \delta(\mu' + a)e^{-j2\pi \mu' t} d\mu'
\]

\[
= e^{j2\pi at} \quad \text{Take Fourier Trans. of both sides}
\]

\[
\mathcal{F}\{\mathcal{F}^{-1}\{\delta(\mu - a)\}\}\] = \mathcal{F}\{e^{j2\pi at}\}

\[
\delta(\mu - a) = \mathcal{F}\{e^{j2\pi at}\}
\]
Convolution and Fourier Trans.

\[ f(t) \ast h(t) = \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau \]

\[ \mathcal{F} \{ f(t) \ast h(t) \} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\tau) h(t - \tau) d\tau \right] e^{-j2\pi\mu t} dt \]

\[ = \int_{-\infty}^{\infty} f(\tau) \left[ \int_{-\infty}^{\infty} h(t - \tau) e^{-j2\pi\mu t} dt \right] d\tau \]

\[ = \int_{-\infty}^{\infty} f(\tau) H(\mu) e^{-j2\pi\mu t} d\tau \]

\[ = H(\mu) \int_{-\infty}^{\infty} f(\tau) e^{-j2\pi\mu t} d\tau = H(\mu) F(\mu) \]
• Convolution in time domain is multiplication in frequency domain

\[
f(t) \ast h(t) \iff H(\mu)F(\mu)
\]

• Multiplication in time domain is convolution in frequency domain

\[
f(t)h(t) \iff H(\mu) \ast F(\mu)
\]
Sampling

- We can sample continuous function $f(t)$ by multiplication with an impulse train

$$\tilde{f}(t) = f(t)s_{\Delta T}(t)$$

$$= \sum_{n=-\infty}^{\infty} f(t)\delta(t - n\Delta T)$$

$$f_k = f(k\Delta T)$$

$$= \int_{-\infty}^{\infty} f(t)\delta(t - k\Delta T)dt$$
Fourier trans. of sampled func.

\[ \tilde{F}(\mu) = \mathcal{F} \{ f(t) s_{\Delta T}(t) \} \]

\[ = F(\mu) \ast S(\mu) \]

\[ = \frac{1}{\Delta T} \int_{-\infty}^{\infty} F(\tau) \sum_{n=-\infty}^{\infty} \delta \left( \mu - \tau - \frac{n}{\Delta T} \right) d\tau \]

\[ = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} F(\tau) \delta \left( \mu - \tau - \frac{n}{\Delta T} \right) d\tau \]

\[ = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F \left( \mu - \frac{n}{\Delta T} \right) \]

• What does this mean?
• Fourier transform of band-limited signal

• Over-sampling

• Critically-sampling

• Under-sampling
Sampling theorem

- When can we recover $f(t)$ from its sampled version?
  - $f(t)$ has to be band-limited
  - If we can isolate a single copy of $F(\mu)$ from the Fourier transform of the sampled signal.

\[
F(\mu) = 0 \quad \forall \mu > \mu_{\text{max}}
\]

where $\mu_{\text{max}} < \infty$

\[
\frac{1}{\Delta T} > 2\mu_{\text{max}}
\]
Function recovery from sample

\[ F(\mu) = H(\mu) \tilde{F}(\mu) \]
Two-dimensional Fourier Transform Pair

\[ F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) e^{-j2\pi(\mu t + \nu z)} \, dt \, dz \]

\[ f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) e^{j2\pi(\mu t + \nu z)} \, d\mu \, d\nu \]
Fourier transform of 2D box

\[ F(\mu, \nu) = ATZ \begin{bmatrix} \frac{\sin(\pi \mu T)}{\pi \mu T} \\ \frac{\sin(\pi \nu Z)}{\pi \nu Z} \end{bmatrix} \]

**FIGURE 4.13** (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the \( t \)-axis, so the spectrum is more “contracted” along the \( \mu \)-axis. Compare with Fig. 4.4.
2D impulse function

\[ \delta(t, z) = \begin{cases} \infty & \text{if } t = z = 0 \\ 0 & \text{otherwise} \end{cases} \]

- Properties
  - Unit area
    \[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t, z) \, dt \, dz = 1 \]
  - Sifting
    \[ \int_{-\infty}^{\infty} f(t, z) \delta(t - t_o, z - z_o) \, dt = f(t_o, z_o) \]
2D sampling

• 2D impulse train as sampling function

\[ s_{\Delta T \Delta Z}(t, z) = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} \delta(t - m \Delta T, z - n \Delta Z) \]

• Sampling theorem
  – Band-limited
  \[ F(\mu, \nu) = 0 \text{ for } \mu > \mu_{max} \text{ or } \nu > \nu_{max} \]
  – Sampling rate limits
  \[ \frac{1}{\Delta T} > 2\mu_{max} \quad \frac{1}{\Delta Z} > 2\nu_{max} \]
Aliasing

- What happens if a band-limited function is sampled at a rate less than the Nyquist frequency?
  - High-frequency components of original signal appear as if they are low-frequency components of the sampled function
  - Alias: false-identity
Aliasing example

\[ f(t) = \sin(\pi t) \]

Figure: Sampling rate less than Nyquist rate

**Period** = 2, **Frequency** = 0.5  
**Nyquist rate** = \(2 \times 0.5 = 1\)

Sampling rate must be strictly greater than the Nyquist rate. What happens if we sample this signal at exactly the Nyquist rate?
Aliasing in images

Over-sampled

Footprint of an ideal lowpass (box) filter

Under-sampled Aliasing

Aliasing example

- Digitizing a checkerboard pattern with 96 x 96 sample array.
  - We can resolve squares that have sides one pixel long or longer.

**FIGURE 4.16** Aliasing in images. In (a) and (b), the lengths of the sides of the squares are 16 and 6 pixels, respectively, and aliasing is visually negligible. In (c) and (d), the sides of the squares are 0.9174 and 0.4798 pixels, respectively, and the results show significant aliasing. Note that (d) masquerades as a “normal” image.
Aliasing in images

- Images always have finite extent (duration) so aliasing is always present.
- Effects of aliasing can be reduced by slightly defocusing the scene to be digitized.
  - This has to be done before the image is sampled!
- Resampling a digital image can also cause aliasing.
  - Blurring (averaging) helps reduce these effects
Alising due to image shrinking

**FIGURE 4.17** Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a $3 \times 3$ averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)
Jagged edges

**FIGURE 4.18** Illustration of jaggies. (a) A $1024 \times 1024$ digital image of a computer-generated scene with negligible visible aliasing. (b) Result of reducing (a) to 25% of its original size using bilinear interpolation. (c) Result of blurring the image in (a) with a $5 \times 5$ averaging filter prior to resizing it to 25% using bilinear interpolation. (Original image courtesy of D. P. Mitchell, Mental Landscape, LLC.)
Inevitable aliasing

- No function of finite duration can be band-limited!!
- Assume we have a band-limited signal of infinite duration. We limit the duration by multiplication with a box function:
  - We already know the Fourier transform of the box function is a sinc function in frequency domain which extends to infinity.
  - Multiplication in time domain is convolution in frequency domain. Therefore, we destroyed the band-limited property of the original signal.
Discrete Fourier Transform

• Fourier transform of sampled data was derived in terms of the transform of the original function:

\[ \tilde{F}(\mu) = \frac{1}{\Delta T} \sum_{n=-\infty}^{\infty} F \left( \mu - \frac{n}{\Delta T} \right) \]

• We want an expression in terms of the sampled function itself. From the definition of the Fourier Transform:

\[ \tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi\mu t} dt \]
\[ \tilde{F}(\mu) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-j2\pi \mu t} dt \]

\[ = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi \mu t} dt \]

\[ = \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \delta(t - n\Delta T) e^{-j2\pi \mu t} dt \]

\[ = \sum_{n=-\infty}^{\infty} f_n e^{-j2\pi \mu n\Delta T} \]
Discrete Fourier Trans. (DFT)

- Notice that the Fourier transform of the discrete signal $f_n$ is continuous and periodic!
- We only need sample one period of the Fourier transform. This is the DFT:

  - Samples taken at $\mu = \frac{m}{M \Delta T}$
  - $m=0,1,...,M-1$

\[
F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M}
\]
Discrete Fourier Transform Pair

\[ F_m = \sum_{n=0}^{M-1} f_n e^{-j2\pi mn/M} \quad m=0,1,\ldots,M-1 \]

\[ f_n = \frac{1}{M} \sum_{m=0}^{M-1} F_m e^{j2\pi mn/M} \quad n=0,1,\ldots,M-1 \]
2D Discrete Fourier Transform

\[ F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \]

\[ f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)} \]

• **Notation**: From now on we will use \( x, y \) and \( u, v \) to denote discrete variables.
• \( f(x, y) \) is a \( M \times N \) digital image
• \( F(u, v) \) is also a 2D matrix of size \( M \times N \). Its elements are complex quantities.
Spatial and frequency intervals

• The entire range of frequencies spanned by the DFT is

\[ u \in \left[ 0, \frac{1}{\Delta T} \right] \text{ and } v \in \left[ 0, \frac{1}{\Delta Z} \right] \]

• The relationship between the spatial and frequency intervals is

\[ \Delta u = \frac{1}{M \Delta T} \text{ and } \Delta v = \frac{1}{N \Delta Z} \]
Periodicity of DFT and 2D DFT

\[ F(u + kM) = \sum_{x=0}^{M-1} f(x) e^{-j2\pi (u+kM)x/M} \]

\[ = \left( \sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \right) e^{-j2\pi kx} \]

\[ = F(u) \]

- Above result holds because \( k \) and \( x \) are integers. This also implies \( f(x) \) obtained by the inverse DFT is periodic! For 2D:
  - \( F(u, v) = F(u + k_1 M, v + k_2 N) \)
  - \( f(x, y) = f(x + k_1 M, y + k_2 N) \)
  - \( k_1 \) and \( k_2 \) integers
Fourier spectrum and phase

- Since the DFT is a complex quantity it can also be expressed in polar coordinates:

\[ F(u, v) = R(u, v) + jI(u, v) \]

\[ F(u, v) = |F(u, v)| e^{j\phi(u,v)} \]

\[ |F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)} \]

\[ \phi(u, v) = \arctan \left[ \frac{I(u, v)}{R(u, v)} \right] \]

4-quadrant arctangent, \textit{atan2} command in \texttt{MATLAB}
Conjugate symmetry

• The Fourier transform of a real function is conjugate symmetric (Section 4.6.6 independent reading)

\[ f(x, y) \text{ real } \iff F^*(u, v) = F(-u, -v) \]

• This means

\[ |F(u, v)| = |F(-u, -v)| \]
\[ \phi(u, v) = -\phi(-u, -v) \]
**DC component**

- Direct current (DC) component means current of frequency = 0. Here we refer to

\[
F(0, 0) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = MN \bar{f}(x, y)
\]

\[
|F(0, 0)| = MN |\bar{f}(x, y)|
\]

- Typically much larger than the non-zero frequency components.
  - Log-scale of the spectrum useful for visualization
Spectrum and phase example
Translation properties

• Translation in space

\[ f(x - x_o, y - y_o) \iff F(u, v)e^{-j2\pi\left(\frac{x_o u}{M} + \frac{y_o v}{N}\right)} \]

• Translation in frequency

\[ f(x, y)e^{j2\pi\left(\frac{x u_o}{M} + \frac{y v_o}{N}\right)} \iff F(u - u_o, v - v_o) \]
Centering the DFT

We want half period (M/2) shift in the frequency domain:

\[ f(x) e^{j2\pi \frac{M/2}{M} x} \iff F(u - \frac{M}{2}) \]

\[ f(x) e^{j\pi x} \iff F(u - \frac{M}{2}) \]

\[ f(x)(-1)^x \iff F(u - \frac{M}{2}) \]
In 2D...

\[ f(x, y)(-1)^{x+y} \iff F(u - M/2, v - N/2) \]

[Diagram with annotations:]
- Four back-to-back periods meet here.
- \( F(u, v) \)

\([\ldots]\) = Periods of the DFT.

\( \square \) = \( M \times N \) data array, \( F(u, v) \).
Rotation property

- Translation in space only affects the phase but not the spectrum of the DFT.
- Rotation in space rotates the DFT (and hence the spectrum) by the same angle.
Phase information

\[ a \quad b \quad c \]

**FIGURE 4.26** Phase angle array corresponding (a) to the image of the centered rectangle in Fig. 4.24(a), (b) to the translated image in Fig. 4.25(a), and (c) to the rotated image in Fig. 4.25(c).

- Phase angle is not intuitive, but it is critical. It determines how the various frequency sinusoids add up. This gives result to shape!
Importance of phase angle

FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.
2D convolution

\[ f(x, y) \ast h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n) \]

• 2D convolution theorem:

\[ f(x, y) \ast h(x, y) \iff F(u, v)H(u, v) \]

• Equivalently:

\[ f(x, y) \ast h(x, y) = \mathcal{F}^{-1} \{ \mathcal{F}\{f(x, y)\}\mathcal{F}\{h(x, y)\}\} \]
Circular convolution

• Implementing the filtering via DFTs and multiplication in the Frequency domain implies an assumption of periodicity
  – This is due to the periodicity property of the DFT we discussed earlier
• Care must be taken in implementing filtering in the frequency domain
• Non-periodic implementation in spatial domain
• Assumes values outside the domain of $f$ and $h$ are
• Note also the length of the output sequence is the sum of the lengths of $f$ and $h$ minus 1

• Periodic implementation
• This is the result that would be obtained if implementing in frequency domain via DFTs
• Incorrect result is obtained because the periods of $f$ and $h$ interfere with each other (wraparound error)
Zero padding solution

• If $f(x)$ has $A$ samples and $h(x)$ has $B$ samples, we avoid the wraparound problem by adding enough zeros to both functions so they both have length $P = A + B - 1$ or longer

• 2D: $f(x,y)$ size $A \times B$, $h(x,y)$ size $C \times D$
  – Pad both to size $P \times Q$ where
  – $P$ is greater than or equal to $A + C - 1$
  – $Q$ is greater than or equal to $B + D - 1$
FIGURE 4.32 (a) A simple image. (b) Result of blurring with a Gaussian lowpass filter without padding. (c) Result of lowpass filtering with padding. Compare the light area of the vertical edges in (b) and (c).

FIGURE 4.33 2-D image periodicity inherent in using the DFT. (a) Periodicity without image padding. (b) Periodicity after padding with 0s (black). The dashed areas in the center correspond to the image in Fig. 4.32(a). (The thin white lines in both images are superimposed for clarity; they are not part of the data.)
Frequency leakage

- When the functions \( f(x,y) \) and/or \( h(x,y) \) are not zero on the image boundaries an abrupt discontinuity is created by zero padding.
  - This is like multiplying the original function with a box filter which in turn implies convolution with a \( \text{sinc} \) function in the frequency domain.
Frequency domain basics

- DC component $F(0,0)$
- Low frequencies - Slowly varying spatial intensities
- High frequencies - Abrupt changes in intensity: edges, noise, etc...
Frequency domain filtering

\[ g(x, y) = \mathcal{F}^{-1} \{ \mathcal{F} \{ f(x, y) \} \mathcal{F} \{ h(x, y) \} \} \]

Lowpass filter \quad Highpass filter
Frequency filtering steps
Impulse response

• Filter $H(u,v)$
• If the input image $f(x, y) = \delta(x, y)$
• The filtered output will be $\mathcal{F}^{-1}\{H(u, v)\}$

$h(x, y) = \mathcal{F}^{-1}\{H(u, v)\}$

is called the impulse response of $H(u,v)$

• All quantities in the discrete implementation are finite: *Finite Impulse Response (FIR) filter*