Lecture 6
Scientific and Data Computing I

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Lecture

- Monday: Start Chapter 7 (Introduction to Time-Stepping)
- Wednesday – first Practicum (attendance taken)
- Monday – second Practicum (no attendance taken)
- Wednesday – Resume Chapter 7
Model Components

• Cash Bond Model
  – B(t) is the cash value at time t
  – r is the rate-of-return (non-volatile).

• Fixed r:
  \[ B(t) = B(0) e^{rt} \]

• Variable r(t):
  \[ B(t) = B(0) \exp \left( \int_0^t r(s) \, ds \right) \]
Questions From Last Time

- Composition Rule
  - Accumulation
  - Hashing

\[
\int_{-\infty}^{b} f(x) \, dx = \sum_{i=0}^{N} \omega_{i} f_{i}
\]

\[
\int_{-\infty}^{b} f(x) \, dx = \int_{-\infty}^{a} f(x) \, dx + \int_{a}^{b} f(x) \, dx
\]
Model Components

- **Cash Bond Model**
  - $B(t)$ is the cash value at time $t$
  - $r$ is the rate-of-return (non-volatile).

- **Origin of this model:**

\[
\frac{dB(t)}{dt} = r(t)B(t)
\]

\[
B(0) = \alpha
\]
Chapter 7: Initial Value Problems

- Canonical example of an initial value problem:

\[ \frac{dy}{dt} = f(t, y), \quad 0 < t \]

\[ y(0) = \alpha \]
Two Perspectives

• Differentiation

\[ \frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{dt} \]

• Integration

\[ \int_{t_n}^{t_{n+1}} \frac{dy}{dt} \, dt = y_{n+1} - y_n \]
Euler Method

- **Euler-Forward**

  \[ y_{n+1} = y_n + hf_n \]

- **Euler-Backward**

  \[ y_{n+1} = y_n + hf_{n+1} \]
Differentiation

- **Forward**
  \[ y'(t_j) \approx \frac{y(t_{j+1}) - y(t_j)}{k} \]

- **Centered**
  \[ y'(t_j) \approx \frac{y(t_{j+1}) - y(t_{j-1})}{2k} \]

- **One-Sided**
  \[ y'(t_j) \approx \frac{-y(t_{j+2}) + 4y(t_{j+1}) - 3y(t_j)}{2k} \]
Differentiation (continued)

- Backward Differentiation Formula (BDF-2)

\[ y_{n+1} = \frac{4}{3} y_n - \frac{1}{3} y_{n-1} + \frac{2h}{3} f_{n+1} \]

- BDF-3

\[ y_{n+1} = \frac{18}{11} y_n - \frac{9}{11} y_{n-1} + \frac{2}{11} y_{n-2} + \frac{6h}{11} f_{n+1} \]
Integration

• Fundamental Theorem of Calculus

\[
\int_{t_j}^{t_{j+1}} \frac{dy}{dt} \, dt = \int_{t_j}^{t_{j+1}} f(t, y(t)) \, dt
\]

\[
y(t_{j+1}) = y(t_j) + \int_{t_j}^{t_{j+1}} f(t, y(t)) \, dt
\]
Integration (continued)

• Example Scheme Using The Trapezoidal Rule:

\[
y(t_{j+1}) - y(t_j) = \frac{k}{2}[f(t_{j+1}, y(t_{j+1})) + f(t_j, y(t_j))] + \mathcal{O}(k^3)
\]

\[
y_{j+1} = y_j + \frac{k}{2}(f_{j+1} + f_j), \text{ for } j = 0, 1, 2, \ldots, M - 1
\]
Integration (continued)

• Right Box

\[ \int_{t_j}^{t_{j+1}} f(x) \, dx = k f(t_{j+1}) + O(k^2) \]

• Left Box

\[ \int_{t_j}^{t_{j+1}} f(x) \, dx = k f(t_j) + O(k^2) \]
Integration (continued)

- Midpoint

\[
\int_{t_j}^{t_{j+1}} f(x) \, dx = 2k f(t_j) + \frac{k^3}{3} f''(\eta_i)
\]

- Trapezoidal

\[
\int_{t_j}^{t_{j+1}} f(x) \, dx = \frac{k}{2} (f(t_j) + f(t_{j+1})) - \frac{k^3}{12} f''(\eta_j)
\]
Integration (continued)

• Simpson’s Method

\[
\int_{t_{j-1}}^{t_{j+1}} f(x) \, dx = \frac{k}{3} \left( f(t_{j+1}) + 4f(t_j) + f(t_{j-1}) \right) - \frac{k^5}{90} f^{(5)}(\eta_j)
\]
Adams Family

- Adams-Bashforth (2\textsuperscript{nd} Order)

\[ y_{n+1} = y_n + \frac{3h}{2} f_n - \frac{h}{2} f_{n-1} \]

- Adams-Bashforth (3\textsuperscript{rd} Order)

\[ y_{n+1} = y_n + \frac{23h}{12} f_n - \frac{4h}{3} f_{n-1} + \frac{5h}{12} f_{n-2} \]
Adams Family (continued)

- Adams-Moulton (2\textsuperscript{nd} Order)

\[ y_{n+1} = y_n + \frac{h}{2} (f_{n+1} + f_n) \]

- Adams-Moulton (3\textsuperscript{rd} Order)

\[ y_{n+1} = y_n + \frac{5h}{12} f_{n+1} + \frac{2h}{3} f_n - \frac{h}{12} f_{n-1} \]
For Next Time

• Read Chapter 7 (for time-stepping)
• Look Ahead Chapter 4 and 9 (for volatility)
• Start working on Practicums
• Start working on Homework 1