Lecture 23
Scientific and Data Computing I

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Lecture

• Wednesday – Constrained Optimization and Practicum
• Final Project: Due December 15 at 5pm MT.
Constrained Problem

- General form

\[
\min_{x \in \Omega} \phi(x), \quad \text{where} \\
\Omega = \{x \in \mathbb{R}^n | c_i(x) = 0, \ i \in \mathcal{E}, \ c_i(x) \geq 0, \ i \in \mathcal{I}\}.
\]

- **Equality constraints**: reducing space; algebraic; domain \(\Omega\) has empty interior.

- **Inequality constraints**: combinatorial; if \(\mathcal{E}\) empty then domain \(\Omega\) can have nonempty interior.

- **Active set**

\[
\mathcal{A}(x) = \mathcal{E} \cup \{i \in \mathcal{I} | c_i(x) = 0\}.
\]

Consider problems where \(\mathcal{A}(x^*)\) is nonempty.
KKT Conditions For A Minimum

- Assume constraint qualification.
- Lagrangian

\[ \mathcal{L}(x, \lambda) = \phi(x) - \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x). \]

- KKT conditions

\[
\begin{align*}
\nabla_x \mathcal{L}(x^*, \lambda^*) &= 0, \\
    c_i(x^*) &= 0, \quad \forall i \in \mathcal{E}, \\
    c_i(x^*) &\geq 0, \quad \forall i \in \mathcal{I}, \\
    \lambda_i^* &\geq 0, \quad \forall i \in \mathcal{I}, \\
    \lambda_i^* c_i(x^*) &= 0, \quad \forall i \in \mathcal{E} \cup \mathcal{I}.
\end{align*}
\]
Active Set Methods

- Assuming solution is on $\partial \Omega$, search for the optimum along the boundary.
- For inequality constraints, keep track of $\mathcal{A}(x_k)$, shuffling constraints in and out of the active set.
- e.g. quadratic programming (QP): quadratic objective function subject to linear inequality constraints.
- Sequential quadratic programming (SQP): At each iteration solve QP for search direction $p_k$ at $(x_k, \lambda_k)$

$$\min_p \frac{1}{2} p^T W_k p + \nabla \phi(x_k)^T p,$$
$$c_i(x_k) + \nabla c_i(x_k)^T p = 0, \quad i \in \mathcal{E}, \quad c_i(x_k) + \nabla c_i(x_k)^T p \geq 0, \quad i \in \mathcal{I}.$$
Interior Point and Other Methods

- Penalty methods

\[ \min_x \psi(x, \mu) = \phi(x) + \frac{1}{2\mu} \sum_{i \in \mathcal{E}} c_i^2(x), \]

where \( \mu \downarrow 0 \).

- Barrier methods

\[ \min_x \psi(x, \mu) = \phi(x) - \mu \sum_{i \in \mathcal{I}} \log c_i(x), \]

where \( \mu \downarrow 0 \).

- Augmented Lagrangian

\[ \min_x \psi(x, \lambda, \mu) = \phi(x) - \sum_{i \in \mathcal{E}} \lambda_i c_i(x) + \frac{1}{2\mu} \sum_{i \in \mathcal{E}} c_i^2(x). \]

Given estimates \( \lambda_k, \mu_k \), solve the unconstrained minimization problem for \( x = x_{k+1} \), then update the multipliers to \( \lambda_{k+1}, \mu_{k+1} \).
For Next Time

• There is no next time …. 😊
• Finish working on Final Project. Due December 15 at 5pm MT.

• Now on to the Practicum …