Using Theorem 1.8, we can extend our discussion to **multinomial coefficients**, that is, to the coefficients that arise in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$. The multinomial coefficient of the term $x_1^{r_1} \cdot x_2^{r_2} \cdot \ldots \cdot x_k^{r_k}$ in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ is

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! \cdot r_2! \cdot \ldots \cdot r_k!}$$

EXAMPLE 1.23

What is the coefficient of $x_1^3 x_2 x_3^2$ in the expansion of $(x_1 + x_2 + x_3)^6$?

Solution Substituting $n = 6$, $r_1 = 3$, $r_2 = 1$, and $r_3 = 2$ into the preceding formula, we get

$$\frac{6!}{3! \cdot 1! \cdot 2!} = 60$$

SES

1.1. An operation consists of two steps, of which the first can be made in n_1 ways. If the first step is made in the ith way, the second step can be made in n_{2i} ways.†

(a) Use a tree diagram to find a formula for the total number of ways in which the total operation can be made.

(b) A student can study 0, 1, 2, or 3 hours for a history test on any given day. Use the formula obtained in part (a) to verify that there are 13 ways in which the student can study at most 4 hours for the test on two consecutive days.

1.2. With reference to Exercise 1.1, verify that if n_{2i} equals the constant n_2, the formula obtained in part (a) reduces to that of Theorem 1.1.

1.3. With reference to Exercise 1.1, suppose that there is a third step, and if the first step is made in the ith way and the second step in the jth way, the third step can be made in n_{3ij} ways.

(a) Use a tree diagram to verify that the whole operation can be made in

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_{2i}} n_{3ij}$$

different ways.

(b) With reference to part (b) of Exercise 1.1, use the formula of part (a) to verify that there are 32 ways in which the student can study at most 4 hours for the test on three consecutive days.

1.4. Show that if n_{2i} equals the constant n_2 and n_{3ij} equals the constant n_3, the formula of part (a) of Exercise 1.3 reduces to that of Theorem 1.2.

1.5. In a two-team basketball play-off, the winner is the first team to win m games.

(a) Counting separately the number of play-offs requiring $m, m + 1, \ldots,$ and $2m - 1$ games, show that the total number of different outcomes (sequences of wins and losses by one of the teams) is

$$2\left[\binom{m-1}{m-1} + \binom{m}{m-1} + \cdots + \binom{2m-2}{m-1}\right]$$

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†The use of double subscripts is explained in Appendix A.

(b) How many different outcomes are there in a "2 out of 3" play-off, a "3 out of 5" play-off, and a "4 out of 7" play-off?

1.6. When n is large, $n!$ can be approximated by means of the expression

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

called **Stirling's formula**, where e is the base of natural logarithms. (A derivation of this formula may be found in the book by W. Feller cited among the references at the end of this chapter.)

(a) Use Stirling's formula to obtain approximations for 10! and 12!, and find the percentage errors of these approximations by comparing them with the exact values given in Table VII.

(b) Use Stirling's formula to obtain an approximation for the number of 13-card bridge hands that can be dealt with an ordinary deck of 52 playing cards.

1.7. Using Stirling's formula (see Exercise 1.6) to approximate $2n!$ and $n!$, show that

$$\frac{\binom{2n}{n}\sqrt{\pi n}}{2^{2n}} \approx 1$$

1.8. In some problems of **occupancy theory** we are concerned with the number of ways in which certain *distinguishable* objects can be distributed among individuals, urns, boxes, or cells. Find an expression for the number of ways in which r *distinguishable* objects can be distributed among n cells, and use it to find the number of ways in which three different books can be distributed among the 12 students in an English literature class.

1.9. In some problems of occupancy theory we are concerned with the number of ways in which certain *indistinguishable* objects can be distributed among individuals, urns, boxes, or cells. Find an expression for the number of ways in which r *indistinguishable* objects can be distributed among n cells, and use it to find the number of ways in which a baker can sell five (indistinguishable) loaves of bread to three customers. (*Hint*: We might argue that L|LLL|L represents the case where the three customers buy one loaf, three loaves, and one loaf, respectively, and that LLLL||L represents the case where the three customers buy four loaves, none of the loaves, and one loaf. Thus, we must look for the number of ways in which we can arrange the five L's and the two vertical bars.)

1.10. In some problems of occupancy theory we are concerned with the number of ways in which certain *indistinguishable* objects can be distributed among individuals, urns, boxes, or cells with at least one in each cell. Find an expression for the number of ways in which r *indistinguishable* objects can be distributed among n cells with at least one in each cell, and rework the numerical part of Exercise 1.9 with each of the three customers getting at least one loaf of bread.

1.11. Construct the seventh and eighth rows of Pascal's triangle and write the binomial expansions of $(x+y)^6$ and $(x+y)^7$.

1.12. Prove Theorem 1.11 by expressing all the binomial coefficients in terms of factorials and then simplifying algebraically.

1.13. Expressing the binomial coefficients in terms of factorials and simplifying algebraically, show that

(a) $\binom{n}{r} = \frac{n-r+1}{r}\cdot\binom{n}{r-1}$;

(b) $\binom{n}{r} = \frac{n}{n-r} \cdot \binom{n-1}{r}$;

(c) $n\binom{n-1}{r} = (r+1)\binom{n}{r+1}$.

1.14. Substituting appropriate values for x and y into the formula of Theorem 1.9, show that

(a) $\sum_{r=0}^{n} \binom{n}{r} = 2^n$;

(b) $\sum_{r=0}^{n} (-1)^r \binom{n}{r} = 0$;

(c) $\sum_{r=0}^{n} \binom{n}{r}(a-1)^r = a^n$.

1.15. Repeatedly applying Theorem 1.11, show that

$$\binom{n}{r} = \sum_{i=1}^{r+1} \binom{n-i}{r-i+1}$$

1.16. Use Theorem 1.12 to show that

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$$

1.17. Show that $\sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}$ by setting $x = 1$ in Theorem 1.9, then differentiating the expressions on both sides with respect to y, and finally substituting $y = 1$.

1.18. Rework Exercise 1.17 by making use of part (a) of Exercise 1.14 and part (c) of Exercise 1.13.

1.19. If n is not a positive integer or zero, the binomial expansion of $(1+y)^n$ yields, for $-1 < y < 1$, the infinite series

$$1 + \binom{n}{1}y + \binom{n}{2}y^2 + \binom{n}{3}y^3 + \cdots + \binom{n}{r}y^r + \cdots$$

where $\binom{n}{r} = \frac{n(n-1)\cdot\ldots\cdot(n-r+1)}{r!}$ for $r = 1, 2, 3, \ldots$. Use this **generalized definition of binomial coefficients** (which agrees with the one on page 11 for positive integral values of n) to evaluate

(a) $\binom{\frac{1}{2}}{4}$ and $\binom{-3}{3}$;

(b) $\sqrt{5}$ writing $\sqrt{5} = 2(1+\frac{1}{4})^{1/2}$ and using the first four terms of the binomial expansion of $(1+\frac{1}{4})^{1/2}$.

1.20. With reference to the generalized definition of binomial coefficients in Exercise 1.19, show that

(a) $\binom{-1}{r} = (-1)^r$;

(b) $\binom{-n}{r} = (-1)^r \binom{n+r-1}{r}$ for $n > 0$.

1.21. Find the coefficient of $x^2y^3z^3$ in the expansion of $(x + y + z)^8$.

1.22. Find the coefficient of $x^3y^2z^3w$ in the expansion of $(2x + 3y - 4z + w)^9$.

1.23. Show that

$$\binom{n}{n_1, n_2, \ldots, n_k} = \binom{n-1}{n_1 - 1, n_2, \ldots, n_k} + \binom{n-1}{n_1, n_2 - 1, \ldots, n_k} + \cdots + \binom{n-1}{n_1, n_2, \ldots, n_k - 1}$$

by expressing all these multinomial coefficients in terms of factorials and simplifying algebraically.

HE THEORY IN PRACTICE

Applications of the preceding theory of combinatorial methods and binomial coefficients are quite straightforward, and a variety of them has been given in Sections 1.2 and 1.3. The following examples illustrate further applications of this theory.

EXAMPLE 1.24

An assembler of electronic equipment has 20 integrated-circuit chips on her table, and she must solder three of them as part of a larger component. In how many ways can she choose the three chips for assembly?

Solution Using Theorem 1.6, we obtain the result

$$_{20}P_3 = 20!/17! = 20 \cdot 19 \cdot 18 = 6,840$$

EXAMPLE 1.25

A lot of manufactured goods, presented for sampling inspection, contains 16 units. In how many ways can 4 of the 16 units be selected for inspection?

Solution According to Theorem 1.7,

$$\binom{16}{4} = 16!/4! = 16 \cdot 15 \cdot 14 \cdot 13/4 \cdot 3 \cdot 2 \cdot 1 = 1,092 \text{ ways}$$

ED EXERCISES SECS. 1.1–1.4

1.24. On August 31 there are five wild-card terms in the American League that can make it to the play-offs, and only two will win spots. Draw a tree diagram which shows the various possible play-off wild-card teams.

1.25. A thermostat will call for heat 0, 1, or 2 times a night. Construct a tree diagram to show that there are 10 different ways that it can turn on the furnace for a total of 6 times over 4 nights.

1.26. There are four routes, A, B, C, and D, between a person's home and the place where he works, but route B is one-way, so he cannot take it on the way to work, and route C is one-way, so he cannot take it on the way home.

(a) Draw a tree diagram showing the various ways the person can go to and from work.

(b) Draw a tree diagram showing the various ways he can go to and from work without taking the same route both ways.

1.27. A person with \$2 in her pocket bets \$1, even money, on the flip of a coin, and she continues to bet \$1 as long as she has any money. Draw a tree diagram to show the various things that can happen during the first four flips of the coin. After the fourth flip of the coin, in how many of the cases will she be

(a) exactly even;

(b) exactly \$2 ahead?

1.28. Suppose that in a baseball World Series (in which the winner is the first team to win four games) the National League champion leads the American League champion three games to two. Construct a tree diagram to show the number of ways in which these teams may win or lose the remaining game or games.

1.29. The pro at a golf course stocks two identical sets of women's clubs, reordering at the end of each day (for delivery early the next morning) if and only if he has sold them both. Construct a tree diagram to show that if he starts on a Monday with two sets of the clubs, there are altogether eight different ways in which he can make sales on the first two days of that week.

1.30. Counting the number of outcomes in games of chance has been a popular pastime for many centuries. This was of interest not only because of the gambling that was involved, but also because the outcomes of games of chance were often interpreted as divine intent. Thus, it was just about a thousand years ago that a bishop in what is now Belgium determined that there are 56 different ways in which three dice can fall *provided one is interested only in the overall result and not in which die does what*. He assigned a virtue to each of these possibilities and each sinner had to concentrate for some time on the virtue that corresponded to his cast of the dice.

(a) Find the number of ways in which three dice can all come up with the same number of points.

(b) Find the number of ways in which two of the three dice can come up with the same number of points, while the third comes up with a different number of points.

(c) Find the number of ways in which all three of the dice can come up with a different number of points.

(d) Use the results of parts (a), (b), and (c) to verify the bishop's calculations that there are altogether 56 possibilities.

1.31. If the NCAA has applications from six universities for hosting its intercollegiate tennis championships in 1998 and 1999, in how many ways can they select the hosts for these championships

(a) if they are not both to be held at the same university;

(b) if they may both be held at the same university?

1.32. The five finalists in the Miss Universe contest are Miss Argentina, Miss Belgium, Miss U.S.A., Miss Japan, and Miss Norway. In how many ways can the judges choose

(a) the winner and the first runner-up;

(b) the winner, the first runner-up, and the second runner-up?

1.33. In a primary election, there are four candidates for mayor, five candidates for city treasurer, and two candidates for county attorney.
(a) In how many ways can a voter mark his ballot for all three of these offices?
(b) In how many ways can a person vote if he exercises his option of not voting for a candidate for any or all of these offices?

1.34. A multiple-choice test consists of 15 questions, each permitting a choice of three alternatives. In how many different ways can a student check off her answers to these questions?

1.35. Determine the number of ways in which a distributor can choose 2 of 15 warehouses to ship a large order.

1.36. A carton of 15 light bulbs contains one that is defective. In how many ways can an inspector choose 3 of the bulbs and
(a) get the one that is defective.
(b) not get the one that is defective?

1.37. The price of a European tour includes four stopovers to be selected from among 10 cities. In how many different ways can one plan such a tour
(a) if the order of the stopovers matters;
(b) if the order of the stopovers does not matter?

1.38. In how many ways can a television director schedule a sponsor's six different commercials during the six time slots allocated to commercials during an hour "special"?

1.39. In how many ways can the television director of Exercise 1.38 fill the six time slots for commercials if the sponsor has three different commercials, each of which is to be shown twice?

1.40. In how many ways can the television director of Exercise 1.38 fill the six time slots for commercials if the sponsor has two different commercials, each of which is to be shown three times?

1.41. In how many ways can five persons line up to get on a bus? In how many ways can they line up if two of the persons refuse to follow each other?

1.42. In how many ways can eight persons form a circle for a folk dance?

1.43. How many permutations are there of the letters in the word
(a) "great";
(b) "greet"?

1.44. How many distinct permutations are there of the letters in the word "statistics"? How many of these begin and end with the letter *s*?

1.45. A college team plays 10 football games during a season. In how many ways can it end the season with five wins, four losses, and one tie?

1.46. If eight persons are having dinner together, in how many different ways can three order chicken, four order steak, and one order lobster?

1.47. In Example 1.4 we showed that a true–false test consisting of 20 questions can be marked in 1,048,576 different ways. In how many ways can each question be marked true or false so that
(a) 7 are right and 13 are wrong;
(b) 10 are right and 10 are wrong;
(c) at least 17 are right?

1.48. Among the seven nominees for two vacancies on a city council are three men and four women. In how many ways can these vacancies be filled
(a) with any two of the seven nominees;
(b) with any two of the four women;
(c) with one of the men and one of the women?

1.49. A shipment of 10 television sets includes three that are defective. In how many ways can a hotel purchase four of these sets and receive at least two of the defective sets?

1.50. Ms. Jones has four skirts, seven blouses, and three sweaters. In how many ways can she choose two of the skirts, three of the blouses, and one of the sweaters to take along on a trip?

1.51. How many different bridge hands are possible containing five spades, three diamonds, three clubs, and two hearts?

1.52. Find the number of ways in which one A, three B's, two C's, and one F can be distributed among seven students taking a course in statistics.

1.53. An art collector, who owns 10 paintings by famous artists, is preparing her will. In how many different ways can she leave these paintings to her three heirs?

1.54. A baseball fan has a pair of tickets for six different home games of the Chicago Cubs. If he has five friends who like baseball, in how many different ways can he take one of them along to each of the six games?

1.55. At the end of the day, a bakery gives everything that is unsold to food banks for the needy. If it has 12 apple pies left at the end of a given day, in how many different ways can it distribute these pies among six food banks for the needy?

1.56. With reference to Exercise 1.55, in how many different ways can the bakery distribute the 12 apple pies if each of the six food banks is to receive at least one pie?

1.57. On a Friday morning, the pro shop of a tennis club has 14 identical cans of tennis balls. If they are all sold by Sunday night and we are interested only in how many were sold on each day, in how many different ways could the tennis balls have been sold on Friday, Saturday, and Sunday?

1.58. Rework Exercise 1.57 given that at least two of the cans of tennis balls were sold on each of the three days.

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