Correlation and Covariance

R. F. Riesenfeld
(Based on web slides by
James H. Steiger)

Goals

- Introduce concepts of
  - Covariance
  - Correlation
- Develop computational formulas

Covariance

- ⇨ Variables may change in relation to each other
- ⇨ *Covariance* measures how much the movement in one variable predicts the movement in a corresponding variable

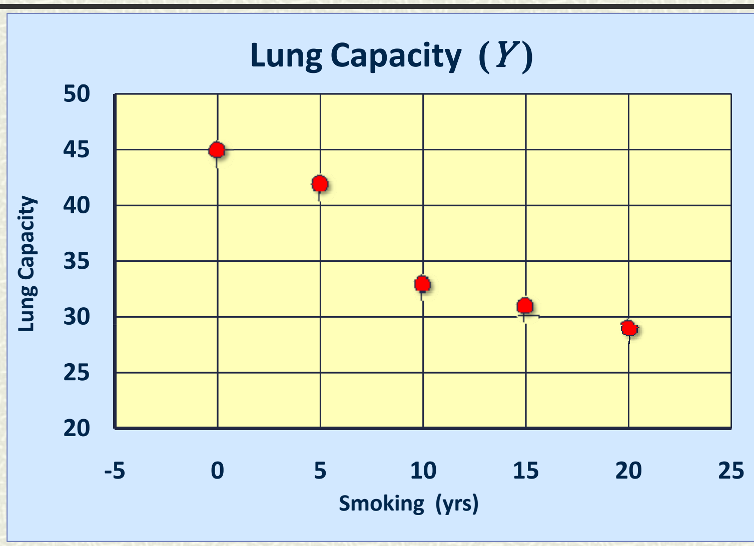
Smoking and Lung Capacity

- ⇨ Example: investigate relationship between *cigarette smoking* and *lung capacity*
- ⇨ Data: sample group response data on smoking habits, *and* measured lung capacities, respectively

Smoking v Lung Capacity Data

N	Cigarettes (X)	Lung Capacity (Y)
1	0	45
2	5	42
3	10	33
4	15	31
5	20	29

Smoking and Lung Capacity



Smoking v Lung Capacity

- Observe that as smoking exposure goes up, corresponding lung capacity goes down
- Variables *covary* inversely
- *Covariance* and *Correlation* quantify relationship

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Covariance

- Variables that *covary* inversely, like smoking and lung capacity, tend to appear on opposite sides of the group means
  - When smoking is above its group mean, lung capacity tends to be below its group mean.
- Average *product of deviation* measures extent to which variables covary, the degree of linkage between them

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The Sample Covariance

- Similar to variance, for theoretical reasons, average is typically computed using (N-1), not N. Thus,

$$S_{xy} = \frac{1}{N-1}\sum_{i=1}^{N}\left(X_i - \bar{X}\right)\left(Y_i - \bar{Y}\right)$$

Calculating Covariance

Cigs (X)	**Lung Cap (Y)**
0	45
5	42
10	33
15	31
20	29
$\bar{X} = 10$	$\bar{Y} = 36$

Calculating Covariance

Cigs (X)	$(X-\bar{X})$	$(X-\bar{X})(Y-\bar{Y})$	$(Y-\bar{Y})$	Cap (Y)
0	-10	-90	9	45
5	-5	-30	6	42
10	0	0	-3	33
15	5	-25	-5	31
20	10	-70	-7	29
		$\sum$= -215		

Covariance Calculation (2)

Evaluation yields,

$$S_{xy} = \frac{1}{4}(-215) = -53.75$$

Covariance under Affine Transformation

Let $L_i = aX_i + b$ and $M_i = cY_i + d$. Then,

$$(\Delta l)_i = a(\Delta x)_i \ , \quad (\Delta m)_i = c(\Delta y)_i \ ,$$

where, $(\Delta u)_i \equiv u_i - \bar{u}$.

Evaluating, in turn, gives,

$$S_{LM} = \frac{1}{N-1}\sum_{i=1}^{N}(\Delta l)_i(\Delta m)_i$$

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Covariance under Affine Transf (2)

Evaluating further,

$$\begin{aligned} S_{LM} &= \frac{1}{N-1}\sum_{i=1}^{N}(\Delta l)_i(\Delta m)_i \\ &= \frac{1}{N-1}\sum_{i=1}^{N}a(\Delta x)_i\, c(\Delta y)_i \\ &= ac\frac{1}{N-1}\sum_{i=1}^{N}(\Delta x)_i(\Delta y)_i \end{aligned}$$

$$\boxed{\therefore\ S_{LM} = acS_{xy}}$$

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(Pearson) Correlation Coefficient r_{xy}

- Like covariance, but uses *Z*-values instead of deviations. Hence, invariant under linear transformation of the raw data.

$$r_{xy} = \frac{1}{N-1}\sum_{i=1}^{N} zx_i zy_i$$

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Alternative (common) Expression

$$r_{xy} = \frac{s_{xy}}{s_x s_y}$$

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Computational Formula 1

$$s_{xy} = \frac{1}{N-1}\left(\sum_{i=1}^{N} X_i Y_i - \frac{\sum_{i=1}^{N} X_i \sum_{i=1}^{N} Y_i}{N} \right)$$

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Computational Formula 2

$$r_{xy} = \frac{N\sum XY - \sum X \sum Y}{\sqrt{\left(N\sum X^2 - \left(\sum X\right)^2\right)\left(N\sum Y^2 - \left(\sum Y\right)^2\right)}}$$

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Table for Calculating r_{xy}

Cigs (X)	X^2	XY	Y^2	Cap (Y)
0	0	0	2025	45
5	25	210	1764	42
10	100	330	1089	33
15	225	465	961	31
20	400	580	841	29
$\sum=$ 50	750	1585	6680	180

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Computing r_{xy} from Table

$$r_{xy} = \frac{5(1585) \; - 50(180)}{\sqrt{\left(5(750 - 50^2)\right)\left(5(6680) - 180^2\right)}}$$

$$= \frac{7925 - 9000}{\sqrt{(3750 - 2500)(33400 - 32400)}}$$

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Computing Correlation

$$r_{xy} = \frac{-1075}{\sqrt{(1250)(1000)}}$$

$$\boxed{r_{xy} = -0.9615}$$

$r_{xy} = -0.96$ Conclusion

- ⇨ r_{xy} = -0.96 implies almost certainty smoker will have diminish lung capacity
- ⇨ Greater smoking exposure implies greater likelihood of lung damage

End

Covariance & Correlation

Notes