Course: CS 5961/6951 *Computational Statistics* Spring 2009
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Lecture Notes: Binomial distribution: expectation and variance

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Definitions:

$$\mathrm{b(k;n,p)} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = \mathrm{E(X)} = \sum_{k=0}^{n} \mathrm{k} \binom{n}{k} p^k (1-p)^{n-k}$$

$$\sigma^2 = \mathrm{Var(X)} = \sum_{k=0}^{n} \mathrm{k}^2 \binom{n}{k} p^k (1-p)^{n-k}$$

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1. As a preliminary step, we observe that the first term in the summation is annihilated by the factor k, because k=0. Therefore, we can drop the first term, and simply begin the summation at k=1,

$$\mathrm{E(X)} = \sum_{k=0}^{n} \mathrm{k} \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \mathrm{k} \binom{n}{k} p^k (1-p)^{n-k}$$

Recalling the recursive relationship for binomial coefficients $\mathrm{k}\binom{\mathrm{n}}{\mathrm{k}} = \mathrm{n}\binom{\mathrm{n-1}}{\mathrm{k-1}}$, we use it to make a substitution,

$$= \sum_{k=1}^{n} \mathrm{n} \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

Now, since n and p are constants, we can pull them out of the summation operator, which yields,

$$= \mathrm{np} \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1}(1-p)^{n-k}$$

Note that the exponent of p^{k-1} above is reduce by 1 to reflect a factor of p has been factored outside the summation.

Changing the summation index to $j=k-1$, we can sum from 0 to $n-1$ instead of 1 to n,

$$= \mathrm{np} \sum_{j=0}^{n-1} \binom{n-1}{j} p^{j}(1-p)^{n-\mathrm{j}-1}$$

If we observe that the exponent n-j-1=((n-1)-j), it is clear that the summation expression is now in canonical form for $\Sigma b(j;n\text{-}1,p)$. That is, the summation takes place over all possible outcomes j for the binomial distribution governing (n-1) Bernoulli trials. Therefore, the summation must simply equal *1*, and we now can conclude that,

$$\boxed{\mu = E(X) = np}$$

This coincides with our intuition on how many heads we ought to expect, on average, in n tosses of a biased coin with probability of success p.

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2. To calculate the variance of the binomial distribution $b(k;n,p)$, we use the convenient formula,

$$\sigma^2 = Var(X) = E(X^2) - \mu^2$$

We have already derived the value of μ above, so we now only need to consider $E(X^2)$,

$$E(X^2) = \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$
$$= \sum_{k=1}^{n} k \left(k \binom{n}{k} \right) p^k (1-p)^{n-k}$$

Again, the recursive formula for binomial coefficients $k\binom{n}{k} = n\binom{n-1}{k-1}$ is used to make a substitution, and we observe, again, that the first term of the summation is *0* because $k=0$ annihilates it. This leads to the following,

$$= \sum_{k=1}^{n} k\left(n\binom{n-1}{k-1}\right)p^k(1-p)^{n-k}$$

In a manner analogous to part 1 where we derived $E(X)$, we move n outside the summation because it is a constant, and we factor out a p from p^k, thus,

$$= np\sum_{k=1}^{n} k\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}$$

Also we perform a similar change of variable $j=k-1$ on the summation index, in a manner analogous to what we did in part 1, so we now have,

$$= np\sum_{j=0}^{n-1}(j+1)\binom{n-1}{j}p^j(1-p)^{(n-1)-j}$$

Distributing the summation over $(j+1)$, we generate two summations,

$$= np\left(\sum_{j=0}^{n-1} j\binom{n-1}{j}p^j(1-p)^{(n-1)-j} + \sum_{j=0}^{n-1}\binom{n-1}{j}p^j(1-p)^{(n-1)-j}\right)$$

This quickly simplifies if we observe that the second summation is 1 because it is a summation over all possibilities, i.e., $\Sigma b(j;n\text{-}1,p)$, and the first summation is exactly the expectation of the binomial distribution for $(n\text{-}1)$ Bernoulli trials. From part 1, we already have established that the first summation is $p(n\text{-}1)$. Thus, we now have,

$$E(X^2) = np\big((n-1)p+1\big)$$

Returning to the earlier formula,

$$Var(X) = E(X^2) - \mu^2$$

$$= np\big((n-1)p+1\big) - \mu^2$$

Recall that $\mu = np$,

$$= np\big((n-1)p+1\big) - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1 - p)$$

$$\boxed{Var(X) = npq}$$

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