

Bresenham's Midpoint Algorithm

Lecture Set 2

CS5600 Computer Graphics
adapted from
Rich Riesenfeld's slides
Spring 2009

Line Characterizations

- Explicit: $y = mx + B$
- Implicit: $F(x, y) = ax + by + c = 0$
- Constant slope: $\frac{\Delta y}{\Delta x} = k$
- Constant derivative: $f'(x) = k$

Line Characterizations - 2

- Parametric: $P(t) = (1-t)P_0 + tP_1$
where, $P(0) = P_0; P(1) = P_1$
- What Else?

Line Characterizations - 2

- Parametric: $P(t) = (1-t)P_0 + tP_1$
where, $P(0) = P_0; P(1) = P_1$
- Intersection of 2 planes
- Shortest path between 2 points
- *Convex hull* of 2 discrete points

Two Line Equations

- Explicit: $y = mx + B$
- Implicit: $F(x, y) = ax + by + c = 0$

Define: $dy = y_1 - y_0$

$dx = x_1 - x_0$

Hence, $y = \left(\frac{dy}{dx} \right) x + B$

From previous

We have, $y = \left(\frac{dy}{dx} \right) x + B$

Hence, $\frac{dy}{dx} x - y + B = 0$

Relating Explicit to Implicit Eq's

Recall, $\frac{dy}{dx}x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$\therefore F(x, y) = (dy)x + (-dx)y + (dx)B = 0$

where, $a = (dy); \quad b = -(dx); \quad c = B(dx)$

Discrete Lines

- Lines vs. Line Segments
- What is a discrete line segment?
 - This is a relatively recent problem
 - How to generate a discrete line?

“Good” Discrete Line

- No gaps in adjacent pixels
- Pixels close to ideal line
- Consistent choices; same pixels in same situations
- Smooth looking
- Even brightness in all orientations
- Same line for $P_0 P_1$ as for $P_1 P_0$
- Double pixels stacked up?

How to Draw a Line?

1. Compute slope
2. Start at on point (x_0, y_0)
3. Increment Δx and draw
 - How to figure this out?

Derive from Line Equation

$$Y = mX + b$$

$$Y_i = mX_i + b$$

$$X_{i+1} = X_i + \Delta X$$

$$Y_{i+1} = mX_{i+1} + b$$

$$\begin{aligned} Y_{i+1} &= m(X_i + \Delta X) + b \\ &= mX_i + m\Delta X + b \\ &= m\Delta X + mX_i + b \\ &= m\Delta X + Y_i \end{aligned}$$

Derive from Line Equation

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$$m = \Delta y / \Delta x$$

$$= (y_1 - y_0) / (x_1 - x_0)$$

$$\Delta y = \Delta x (y_1 - y_0) / (x_1 - x_0)$$

$$Y_{i+1} = Y_i + \Delta Y$$

$$= Y_i + \Delta x (y_1 - y_0) / (x_1 - x_0)$$

$$= Y_i + \Delta X(m)$$

Restricted Form

- Line segment in *first* octant with

$$0 < m < 1$$

- After we derive this, we'll look at the other cases (other octants)

Spring 2006

CS 5600

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DDA: Digital Differential Analyzer

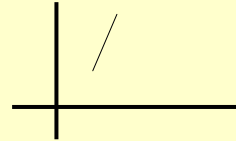
$\Delta y = m \Delta x$

$\Delta y = \Delta x (y_1 - y_0) / (x_1 - x_0)$

- For each Δx ,
Evaluate Δy : $\Delta y = \Delta x (y_1 - y_0) / (x_1 - x_0)$

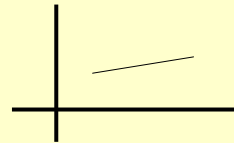
What about slope?

If $|y_1 - y_0| > |x_1 - x_0|$
 increment Y



else

increment X



Incremental Fn Eval

- Recall $f(x_{i+1}) = f(x_i) + \Delta(x_i)$
- Characteristics
 - Fast
 - Cumulative Error
- Need to define $f(x_0)$

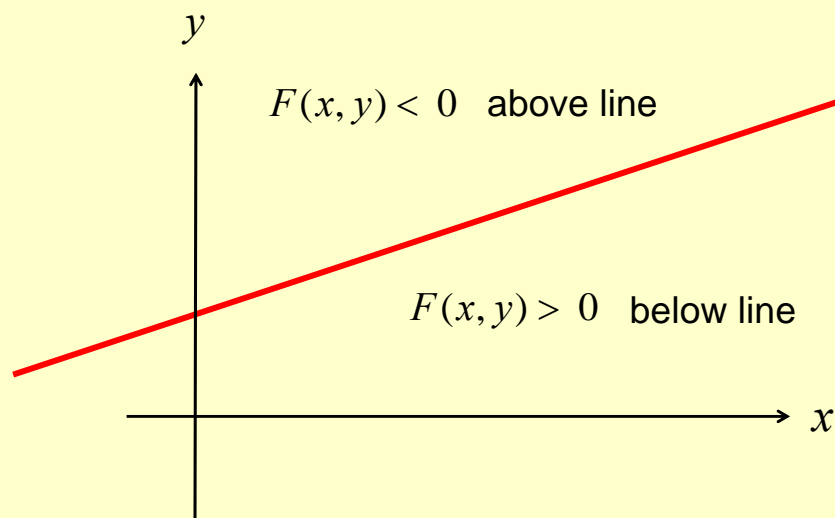
Investigate Sign of F

Verify that

$$F(x, y) = \begin{cases} + & \text{below line} \\ 0 & \text{on line} \\ - & \text{above line} \end{cases}$$

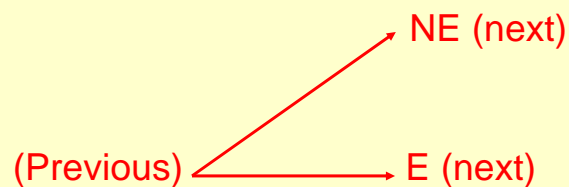
Look at extreme values of y

The Picture



Key to Bresenham Algorithm

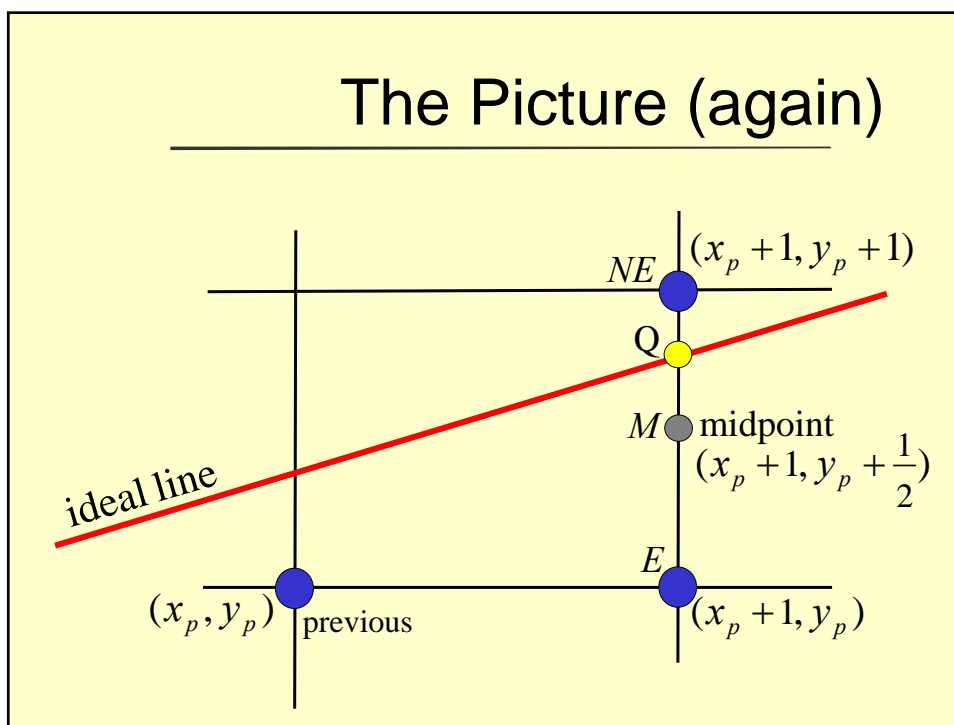
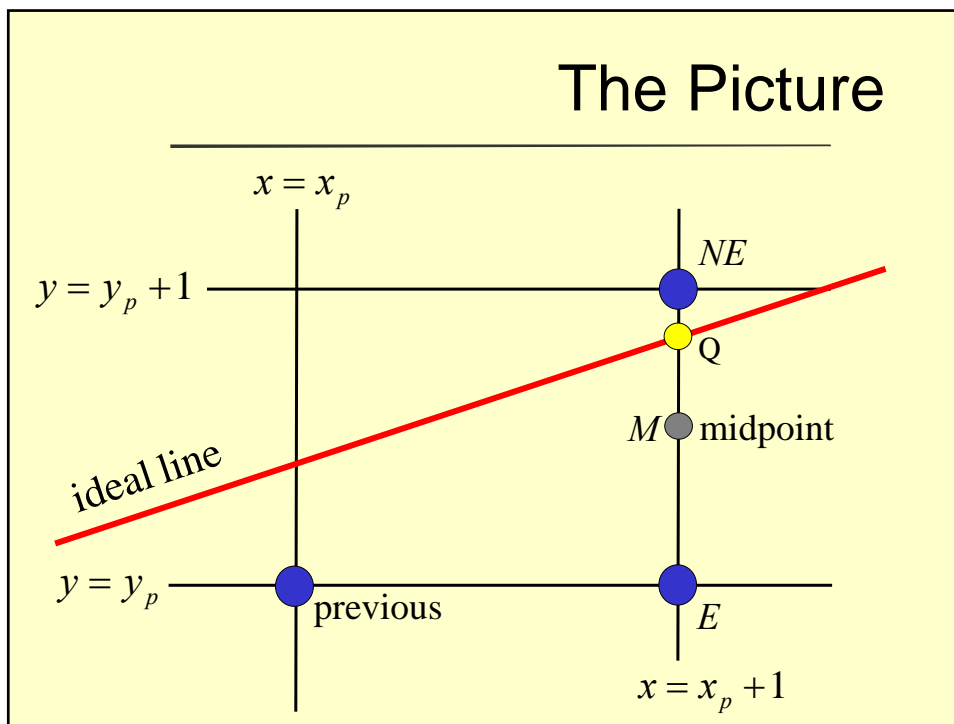
“Reasonable assumptions” have reduced the problem to making a binary choice at each pixel:



Decision Variable d (logical)

Define a logical *decision* variable d

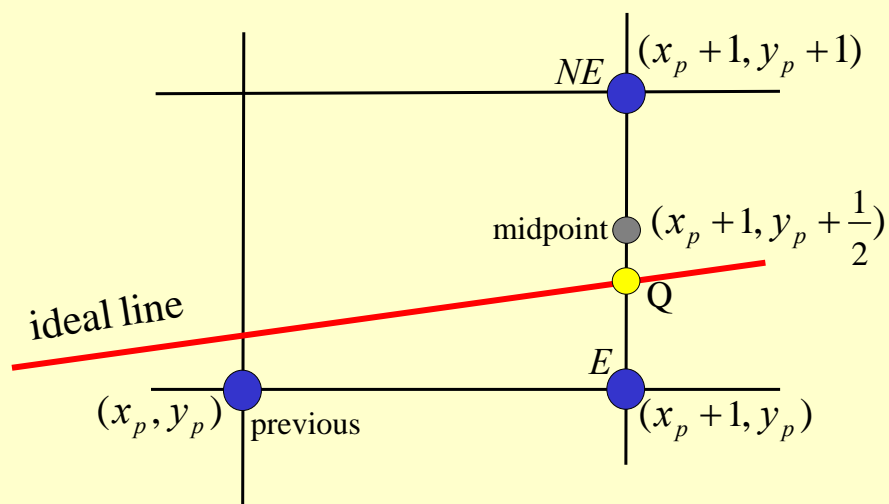
- linear in form
- incrementally updated (with addition)
- tells us whether to go E or NE



Observe the relationships

- Suppose Q is above M , as before.
- Then $F(M) > 0$, M is below the line
- So, $F(M) > 0$ means line is above M ,
- Need to move NE , increase y value

The Picture (again)



Observe the relationships

- Suppose Q is below M , as before.
- Then $F(M) < 0$, implies M is *above* the line
- So, $F(M) < 0$, means line is below M ,
- Need to move to E ; *don't increase y*

$$\underline{M = \text{Midpoint} = (x_p + 1, y_p + \frac{1}{2})}$$

- Want to evaluate at M
- Will use an incremental *decision variable d*

$$d = F(x_p + 1, y_p + \frac{1}{2})$$

- Let, $d = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

How will d be used?

Let,
$$d = a(x_p + 1) + b\left(y_p + \frac{1}{2}\right) + c$$

Therefore,

$$d = \begin{cases} > 0 & \Rightarrow NE \quad (\text{midpoint below ideal line}) \\ < 0 & \Rightarrow E \quad (\text{midpoint above ideal line}) \\ = 0 & \Rightarrow E \quad (\text{arbitrary}) \end{cases}$$

Case E: Suppose E is chosen

- Recall $d_{old} = a(x_p + 1) + b\left(y_p + \frac{1}{2}\right) + c$
- $E \Rightarrow: \quad x \leftarrow x + 1; \quad y \leftarrow y,$
- $\therefore \dots d_{new} = F\left(x_p + 2, y_p + \frac{1}{2}\right)$

$$= a(x_p + 2) + b\left(y_p + \frac{1}{2}\right) + c$$

Case E: Suppose E is chosen

$$d_{new} - d_{old} = \left(a(x_p + 2) + b\left(y_p + \frac{1}{2}\right) + c \right) \\ - \left(a(x_p + 1) + b\left(y_p + \frac{1}{2}\right) + c \right)$$

$$d_{new} = d_{old} + a$$

Review of Explicit to Implicit

Recall, $\frac{dy}{dx}x - y + B = 0$

Or, $(dy)x + (-dx)y + (dx)B = 0$

$\therefore F(x, y) = (dy)x + (-dx)y + (dx)B = 0$

where, $a = (dy)$; $b = -(dx)$; $c = B(dx)$

Case E: $d_{new} = d_{old} + a$

$\Delta_E \equiv$ increment we add if E is chosen.

So, $\Delta_E = a$. But remember that

$a = dy$ (from line equations).

Hence, $F(M)$ is not evaluated explicitly.

We simply add $\Delta_E = a$ to update d for E

Case NE: Suppose NE chosen

Recall $d_{old} = a(x_p + 1) + b(y_p + \frac{1}{2}) + c$

and, $NE \Rightarrow: x \leftarrow x + 1; y \leftarrow y + 1,$

$$\begin{aligned} \therefore d_{new} &= F(x_p + 2, y_p + \frac{3}{2}) \\ &= a(x_p + 2) + b(y_p + \frac{3}{2}) + c \end{aligned}$$

Case NE: Suppose NE

$$\begin{aligned}
 d_{new} - d_{old} &= \\
 &= \left(a(x_p + 2) + b\left(y_p + \frac{3}{2}\right) + c \right) \\
 &\quad - \left(a(x_p + 1) + b\left(y_p + \frac{1}{2}\right) + c \right)
 \end{aligned}$$

$$d_{new} = d_{old} + a + b$$

Case NE: $d_{new} = d_{old} + a + b$

$\Delta_{NE} \equiv$ increment that we add if NE is chosen.

So, $\Delta_{NE} = a + b$. But remember that

$a = dy$, and $b = -dx$ (from line equations).

Hence, $F(M)$ is not evaluated explicitly.

We simply add $\Delta_{NE} = a + b$ to update d for NE

Case NE:

$$\underline{d_{new} = d_{old} + a + b}$$

$$\Delta_{NE} = a + b, \text{ where } a = dy, \text{ and } b = -dx$$

means, we simply add $\Delta_{NE} = a + b$, i.e.,

$$\Delta_{NE} = dy - dx \text{ to update } d \text{ for NE.}$$

Summary

- At each step of the procedure, we must choose between moving E or NE based on the sign of the decision variable d
- Then update according to

$$d \leftarrow \begin{cases} d + \Delta_E, & \text{where } \Delta_E = dy, \text{ or} \\ d + \Delta_{NE}, & \text{where } \Delta_{NE} = dy - dx \end{cases}$$

What is initial value of d ?

- First point is (x_0, y_0)
- First midpoint is $(x_0 + 1, y_0 + \frac{1}{2})$
- What is initial midpoint value?

$$d(x_0 + 1, y_0 + \frac{1}{2}) = F(x_0 + 1, y_0 + \frac{1}{2})$$

What is initial value of d ?

$$\begin{aligned}
 F(x_0 + 1, y_0 + \frac{1}{2}) &= a(x_0 + 1) + b(y_0 + \frac{1}{2}) + c \\
 &= \underbrace{(ax_0 + by_0 + c)}_{F(x_0, y_0)} + \left(a + \frac{b}{2} \right) \\
 &= F(x_0, y_0) + \left(a + \frac{b}{2} \right)
 \end{aligned}$$

What is initial value of d ?

Note, $F(x_0, y_0) = 0$, since (x_0, y_0) is on line.

Hence,

$$F(x_0 + 1, y_0 + \frac{1}{2}) = 0 + a + \frac{b}{2}$$

$$= (dy) - \left(\frac{dx}{2} \right)$$

What Does Factor of 2 x Do ?

- Has the same 0-set

$$2F(x, y) = 2(ax + by + c) = 0$$

- Changes the slope of the plane
- Rotates plane about the 0-set line
- Gets rid of the denominator

What is initial value of d ?

Note, we can clear denominator
and not change line,

$$2F(x_0 + 1, y_0 + \frac{1}{2}) = 2(dy) - dx$$

What is initial value of d ?

$$2F(x, y) = 2(ax + by + c) = 0$$

So, first value of

$$d = 2(dy) - (dx)$$

More Summary

- Initial value $2(dy) - (dx)$
- Case E: $d \leftarrow d + \Delta_E$, where $\Delta_E = 2(dy)$
- Case NE: $d \leftarrow d + \Delta_{NE}$,
where $\Delta_{NE} = 2\{(dy) - (dx)\}$
- Note, all deltas are constants

More Summary

Choose $\begin{cases} E & \text{if } d \leq 0 \\ NE & \text{otherwise} \end{cases}$

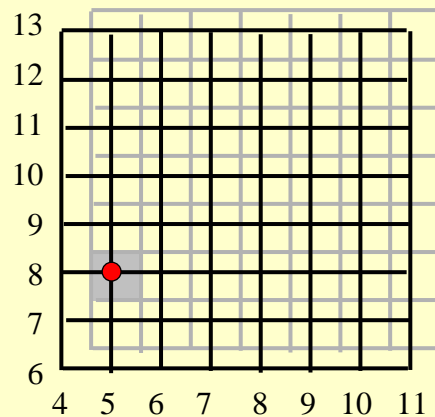
Example

- Line end points:

$$(x_0, y_0) = (5, 8); \quad (x_1, y_1) = (9, 11)$$

- Deltas: $dx = 4$; $dy = 3$

Graph



Example ($dx = 4$; $dy = 3$)

- Initial value of

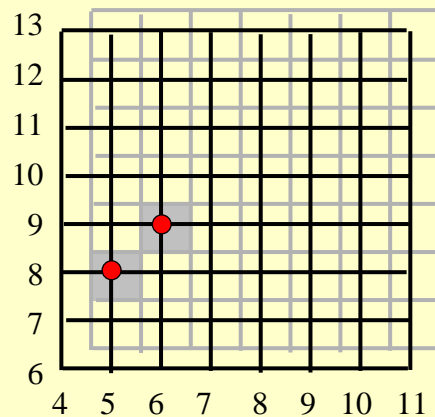
$$\begin{aligned} d(5,8) &= 2(dy) - (dx) \\ &= 6 - 4 = 2 > 0 \end{aligned}$$

$$d = 2 \Rightarrow NE$$

$$d = 2(dy) - (dx)$$

$$\begin{cases} E & \text{if } d \leq 0 \\ NE & \text{otherwise} \end{cases}$$

Graph



Example ($dx=4; dy=3$)

$$d \leftarrow d + \Delta_E, \text{ where } \Delta_E = 2(dy)$$

- Update value of d

$$d \leftarrow d + \Delta_{NE},$$

$$\text{where } \Delta_{NE} = 2\{(dy) - (dx)\}$$

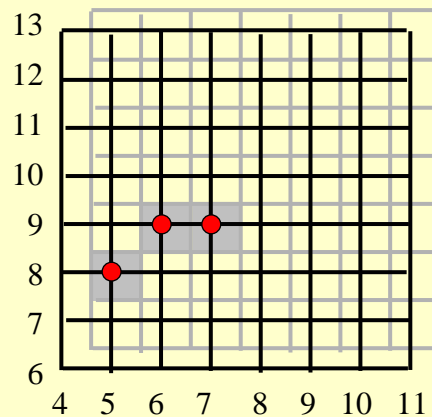
- Last move was NE , so

$$\begin{aligned} \Delta_{NE} &= 2(dy - dx) \\ &= 2(3 - 4) = -2 \end{aligned}$$

$$d = 2 - 2 = 0 \Rightarrow E$$

$$\begin{cases} E & \text{if } d \leq 0 \\ NE & \text{otherwise} \end{cases}$$

Graph



Example ($dx=4$; $dy=3$)

$$d \leftarrow d + \Delta_E, \text{ where } \Delta_E = 2(dy)$$

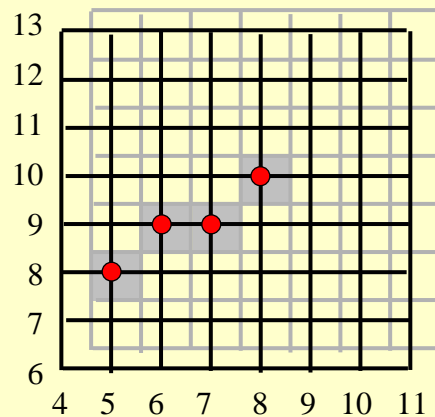
- Previous move was E $d \leftarrow d + \Delta_{NE}$,
where $\Delta_{NE} = 2\{(dy) - (dx)\}$

$$\begin{aligned} \Delta_E &= 2(dy) \\ &= 2(3) = 6 \end{aligned}$$

$$d = 0 + 6 > 0 \Rightarrow NE$$

$$\begin{cases} E & \text{if } d \leq 0 \\ NE & \text{otherwise} \end{cases}$$

Graph



Example ($dx=4$; $dy=3$)

$$d \leftarrow d + \Delta_E, \text{ where } \Delta_E = 2(dy)$$

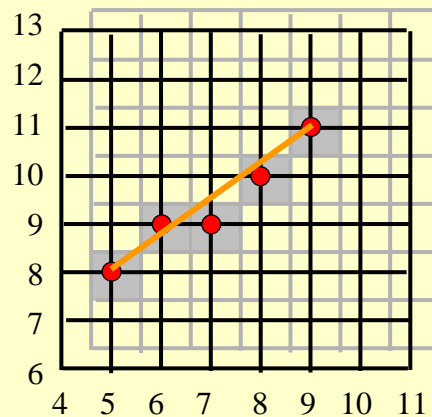
- Previous move was *NE* $d \leftarrow d + \Delta_{NE}$,
where $\Delta_{NE} = 2\{(dy) - (dx)\}$

$$\begin{aligned} \Delta_{NE} &= 2(dy - dx) \\ &= 2(3 - 4) = -2 \end{aligned}$$

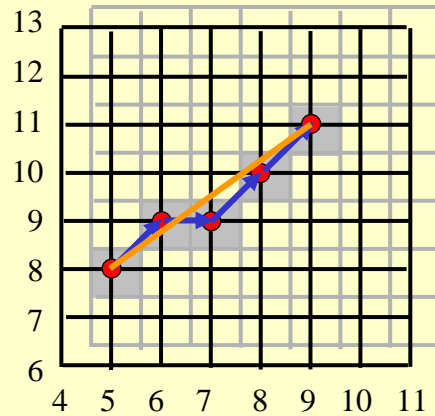
$$d = 6 - 2 = 4 \Rightarrow \mathbf{NE}$$

$$\begin{cases} E & \text{if } d \leq 0 \\ NE & \text{otherwise} \end{cases}$$

Graph



Graph



Meeting Bresenham Criteria

Case 0: $m = 0$; $m = 1 \Rightarrow$ trivial cases

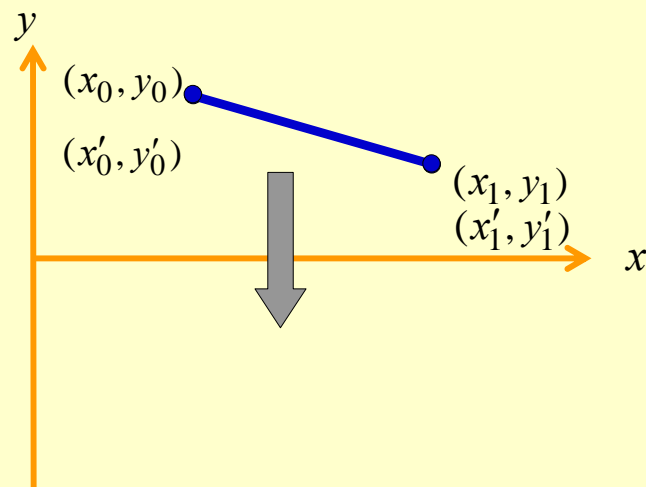
Case 1: $0 > m > -1 \Rightarrow$ flip about x -axis

Case 2: $m > 1 \Rightarrow$ flip about $x = y$

Case 0: Trivial Situations

- $m = 0 \Rightarrow$ horizontal line
- $m = 1 \Rightarrow$ line $y = x$
- Do not need Bresenham

Case 2: Flip about x -axis



Case 2: Flip about x -axis

- Suppose, $0 > m > -1$,
- Flip about x -axis ($y' = -y$):

$$(x'_0, y'_0) = (x_0, -y_0);$$

$$(x'_1, y'_1) = (x_1, -y_1)$$

How do slopes relate?

$$\left. \begin{array}{l} m = \frac{y_1 - y_0}{x_1 - x_0}; \\ m' = \frac{y'_1 - y'_0}{x_1 - x_0} \end{array} \right\} \text{by definition}$$

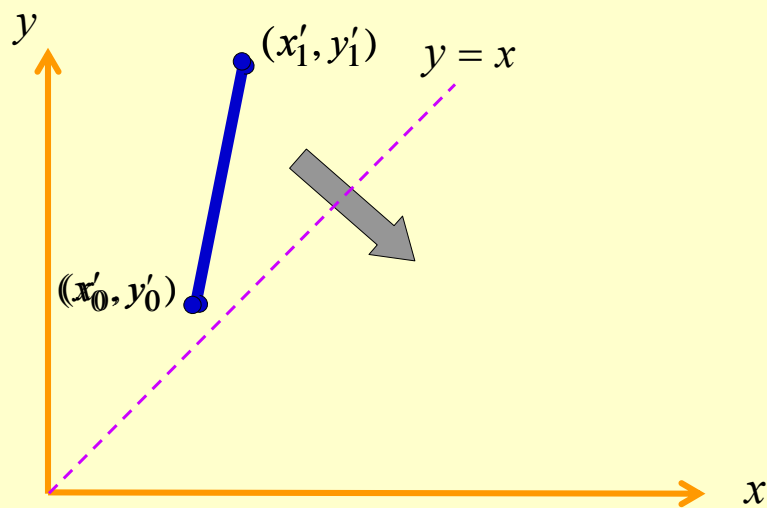
$$\text{Since } y'_i = -y_i, \quad m' = \frac{-y_1 - (-y_0)}{x_1 - x_0}$$

How do slopes relate?

$$\text{i.e., } m' = -\frac{(y_1 - y_0)}{x_1 - x_0}$$
$$m' = -m$$

$$\therefore 0 > m > -1 \Rightarrow 0 < m' < 1$$

Case 3: Flip about line $y = x$



Case 3: Flip about line $y = x$

$$y = mx + B,$$

swap $x \leftrightarrow y$ and prime them ,

$$x' = my' + B,$$

$$my' = x' - B$$

Case 3: $m' = ?$

$$y' = \left(\frac{1}{m}\right)x' - B,$$

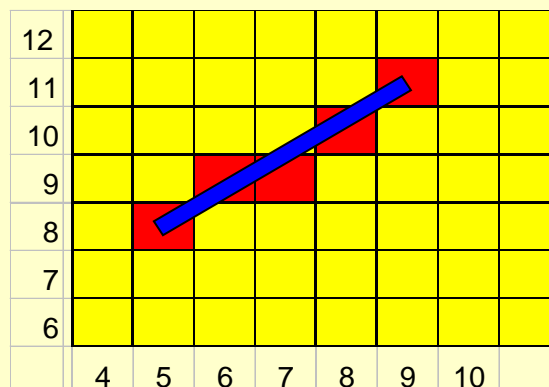
$$\therefore m' = \left(\frac{1}{m}\right) \text{ and,}$$

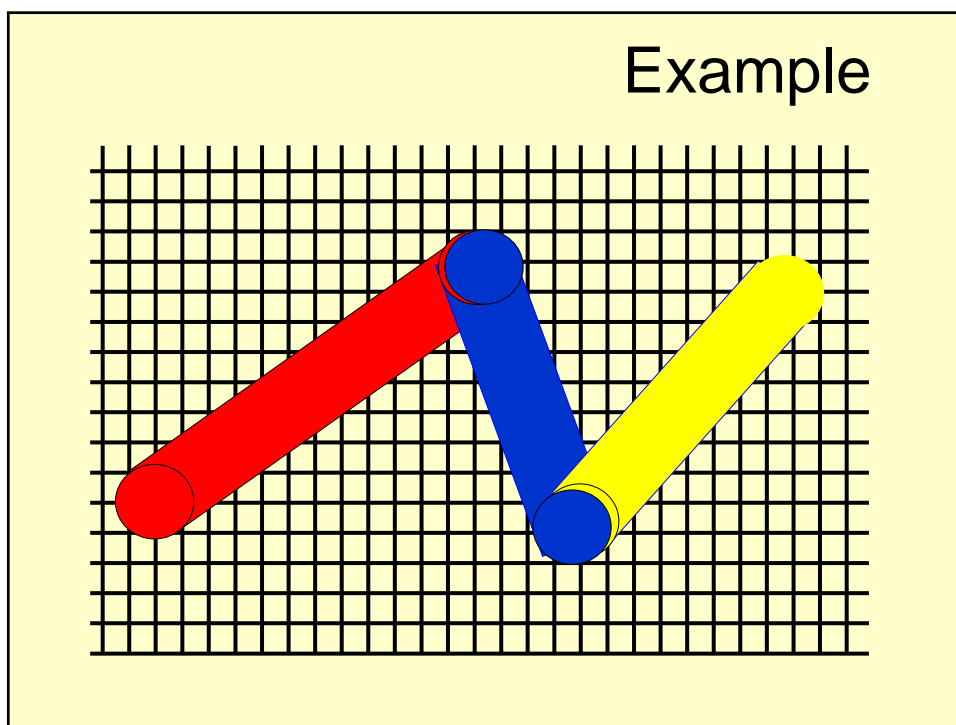
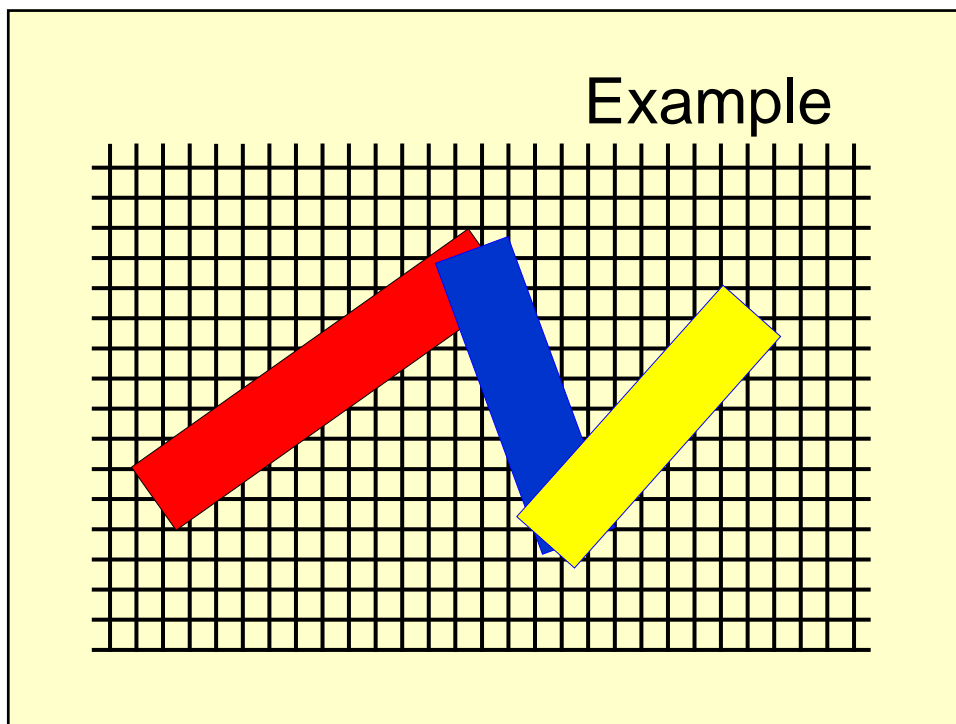
$$m > 1 \Rightarrow 0 < m' < 1$$

More Raster Line Issues

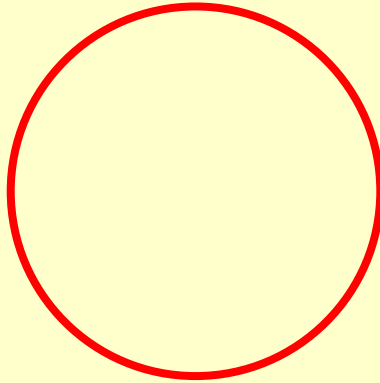
- Fat lines with multiple pixel width
- Symmetric lines
- How should end pt geometry look?
- Generating curves, e.g., circles, etc.
- Jaggies, staircase effect, aliasing...

Pixel Space





Bresenham Circles



The End
Bresenham's Algorithm

Lecture Set 2