Notes: Expectation and Variance

CS 3130 / ECE 3530: Probability and Statistics for Engineers

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Expectation:
The **expectation of a discrete random variable** X taking values $\{a_i\}$ with probability mass function p is given by

$$\mathrm{E}[X] = \sum_i a_i \, P(X = a_i) = \sum_i a_i \, p(a_i).$$

The expectation is the value that you would expect on average if you repeat an experiment many times.

Example: What is the expectation of $X \sim Ber(p)$?

$$E[X] = \sum_{k=0}^{1} k \, p(k) = 0 \cdot (1-p) + 1 \cdot p = p$$

Example: What is the expectation of $X \sim Geo(p)$?

$$E[X] = \sum_{k=1}^{\infty} k \, p(k) = \sum_{k=1}^{\infty} k \, p(1-p)^{k-1} = \frac{1}{p}$$

In-class Exercise: What is the expectation of a six-sided die roll?

$$\mathrm{E}[X] = \frac{1}{6}(1+2+3+4+5+6) = 3.5$$

The **expectation of a continuous random variable** X with probability density function f is given by

$$\mathrm{E}[X] = \int_{-\infty}^{\infty} x \, f(x) \, dx.$$

Example: What is the expectation of $X \sim Exp(\lambda)$?

$$E[X] = \int_{0}^{\infty} x \, f(x) \, dx = \int_{0}^{\infty} x \, \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$$

Example: What is the expectation of $X \sim N(\mu, \sigma^2)$? (from book)

Linearity of expectation: If X and Y are random variables and $a, b \in \mathbb{R}$, then

$$\mathrm{E}[aX + bY] = a \, \mathrm{E}[X] + b \, \mathrm{E}[Y]$$

Example: If we roll 10 dice and sum them up, what is the expected value of the result?
Answer: Let X be one die, and S the sum of 10 dice. Then, $\mathrm{E}[S] = \mathrm{E}[10 \cdot X] = 10 \cdot \mathrm{E}[X] = 35$.

In-class Exercise: Remember that if $X \sim Bin(n, p)$, then X is the sum of n Bernoulli random variables, $X_i \sim Ber(p)$. Use the linearity of expectation to compute $\mathrm{E}[X]$.

$$\mathrm{E}[X] = \mathrm{E}[X_1 + X_2 + \cdots + X_n] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_n] = np$$

Expectation of a function $g : \mathbb{R} \to \mathbb{R}$ of a random variable:
Discrete case:

$$\mathrm{E}[g(X)] = \sum_i g(a_i)\, p(a_i)$$

Continuous case:

$$\mathrm{E}[g(X)] = \int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

Variance:
The **variance** of a random variable X (continuous or discrete) is given by

$$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mathrm{E}[X])^2\right].$$

The variance describes how "spread out" a random variable's distribution is. The **standard deviation**, defined as the square root of the variance, is often a more useful description of the spread (it's in the the same units as $\mathrm{E}[X]$).

Example: The variance of Bernoulli random variable, $X \sim Ber(p)$:

$$\mathrm{Var}(X) = \sum_{k=0}^{1} p(k)(k - \mathrm{E}[X])^2 = (1-p)(0-p)^2 + p(1-p)^2 = (1-p)p^2 + p(1-p)^2 = p(1-p)$$

Example: The variance of the normal distribution, $X \sim N(\mu, \sigma^2)$ (from book).

An equivalent formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2.$$

Variance after a scaling and shift, $a, b \in \mathbb{R}$:

$$\mathrm{Var}(aX + b) = a^2\, \mathrm{Var}(X)$$

In-class exercise: What is the variance of the six-sided die roll?

$$\mathrm{Var}(X) = \sum_{k=1}^{6} \frac{1}{6}k^2 - \mathrm{E}[X]^2 = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36) - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} \approx 2.92$$

$$\text{Standard deviation } = \sqrt{\mathrm{Var}(X)} \approx 1.71$$