LES of Turbulent Flows: Lecture 11
(ME EN 7960-008)

Prof. Rob Stoll
Department of Mechanical Engineering
University of Utah

Spring 2011

Filtering the compressible N-S equations

• What happens when we apply a filter to the compressible N-S equations?

-**<u>Conservation of Mass for compressible flow:</u>** $\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_i}{\partial x_i} = 0$

• filtering each term => $\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \widetilde{\rho u_i}}{\partial x_i} = 0$ → <u>results in an SFS term!</u>

• How can we avoid having a SFS conservation of mass?

-<u>Density weighted filtering:</u>

• Formalized for compressible flow by Favre (Phys. Fluids, 1983) for ensemble statistics, a Favre (or density weighted) filter is defined by:

$$\bar{\phi} = \frac{\widetilde{\rho \phi}}{\tilde{\rho}} \Rightarrow \tilde{\rho}\bar{\phi} = \widetilde{\rho \phi}$$

where we note that as compressibility becomes less important $\bar{\phi} \rightarrow \tilde{\phi}$ and we can show that the conservation of mass becomes:

$$\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho} \bar{u}_i}{\partial x_i} = 0$$

Filtering the compressible N-S equations

• We can use this to write the Favre filtered equations of motion (see Geurts pg 32-35 or Vreman et al. Applied Sci. Res. 1995 for details)

-**Conservation of Mass:** $\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial \tilde{\rho}\bar{u}_i}{\partial x_i} = 0$

-**Conservation of Momentum:**

$$\frac{\partial \tilde{\rho}\bar{u}_i}{\partial t} + \frac{\partial \tilde{\rho}\bar{u}_i\bar{u}_j}{\partial x_j} + \frac{\partial \tilde{p}}{\partial x_i} - \frac{\partial \bar{\sigma}_{ij}}{\partial x_j} = -\frac{\partial \tilde{\rho}\tau_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\tilde{\sigma}_{ij} - \bar{\sigma}_{ij}\right)$$

where the SFS terms are collected on the RHS of the equation and we now have both a resolved ($\bar{\sigma}_{ij}$) and SFS ($\tilde{\sigma}_{ij} - \bar{\sigma}_{ij}$) viscous contribution because $\mu = \mu\left(\bar{T}\right)$ is a function of the Favre filtered temperature and

$$\tilde{\sigma}_{ij} = \left(\frac{2}{Re}\right)\widetilde{\mu\left(T\right)\left(S_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)}$$ => nonlinear viscous stress tensor

$$\bar{\sigma}_{ij} = \left(\frac{2}{Re}\right)\mu\left(\bar{T}\right)\left(\bar{S}_{ij} - \frac{1}{3}\delta_{ij}\frac{\partial \bar{u}_k}{\partial x_k}\right)$$ => "smooth" viscous stress tensor

Recall: strain rate tensor

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

• The SFS stress tensor for the Favre filtered equations is given by

$$\tau_{ij} = \overline{u_i u_j} - \bar{u}_i\bar{u}_j$$

which is obtained from the nonlinear term => $\widetilde{\rho u_i u_j} = \tilde{\rho}\overline{u_i u_j} = \tilde{\rho}\left(\overline{u_i u_j} - \bar{u}_i\bar{u}_j\right)$

Filtering the compressible N-S equations

-**Conservation of total energy:** $\bar{e} \equiv \text{Favre filtered total energy density} = \frac{\tilde{p}}{\gamma - 1} + \frac{1}{2}\tilde{\rho}\bar{u}_i\bar{u}_i$

$$\frac{\partial \bar{e}}{\partial t} + \frac{\partial}{\partial x_j}\left((\bar{e} + \tilde{p})\,\bar{u}_j\right) - \frac{\partial \bar{u}_i \bar{\sigma}_{ij}}{\partial x_j} + \frac{\partial \bar{q}_j}{\partial x_j} = -a_1 - a_2 - a_3 + a_4 + a_5 - a_6$$

where the LHS contains the SFS terms created using the procedure used in Lecture 5 for τ_{ij}

$$a_1 = \bar{u}_i \frac{\partial \tilde{p}\tau_{ij}}{\partial x_j}$$

➔ kinetic energy transferred from resolved to SFSs

$$a_2 = \frac{1}{\gamma - 1}\frac{\partial\left(\widetilde{pu_j} - \tilde{p}\bar{u}_j\right)}{\partial x_j}$$

➔ pressure velocity SFS term (effect of SFS turbulence on the conduction of heat at resolved scales

$$a_3 = \widetilde{p\frac{\partial u_j}{\partial x_j}} - \tilde{p}\frac{\partial \bar{u}_j}{\partial x_j}$$

➔ compressibility effects (vanishes for incompressible)

$$a_4 = \widetilde{\sigma_{ij}\frac{\partial u_i}{\partial x_j}} - \tilde{\sigma}_{ij}\frac{\partial \bar{u}_i}{\partial x_j}$$

➔ conversion of SFS kinetic energy to internal energy by viscous dissipation

$$a_5 = \frac{\partial\left(\tilde{\sigma}_{ij}\bar{u}_i - \bar{\sigma}_{ij}\bar{u}_i\right)}{\partial x_j}$$

➔ SFS viscous stress term

$$a_6 = \frac{\partial\left(\tilde{q}_j - \bar{q}_j\right)}{\partial x_j}$$

➔ SFS heat flux term (Note q_j is the heat flux vector)

Typically assumptions that $\tilde{\sigma}_{ij} - \bar{\sigma}_{ij} \approx 0$ and $\tilde{q}_j - \bar{q}_j \approx 0$ are made eliminating a_5 and a_6 .

Turbulence modeling (alternative strategies)

• So far our discussion of turbulence modeling has centered around separating the flow into resolved and SFSs using a low-pass filtering operation with the goal of reducing the # of degrees of freedom in our numerical solution.

• This is not the only way to accomplish complexity reduction in a turbulent flow. Here we briefly review a few other methods

• **Coherent Vortex Simulations (CVS)**: Farge and Schneider, Flow Turb. Comb. 2001
The turbulent flow field is decomposed into coherent and random components using either a continuous or orthonormal wavelet filter (see the next page for a very brief review of wavelets)

-In the original formulation the separation between coherent and random motions is assumed to be complete with the random part mimicking viscous dissipation.
-Goldstein and Vasilyev (Phys of Fluids, 2004) introduced "**stochastic coherent adaptive LES**", a variation on CVS

- they use the CVS wavelet decomposition but do not assume that the wavelet filter completely eliminates all the coherent motions from the SFSs => the SFS components themselves contain coherent and random components.

Wavelet Decomposition (a brief overview)

- The Coherent Vortex Simulations (CVS) method uses wavelet decomposition.
- Here is a brief overview of wavelets. For a more detailed view see:
  - -Daubechies, 1992 (most recent printing is 2006)
  - -Mallat, 2009 (3rd edition)
  - -Farge, Ann. Rev. Fluid Mech., 1992 (specific to turbulence research)

First: Windowed Fourier Transforms (or Gabor transform)

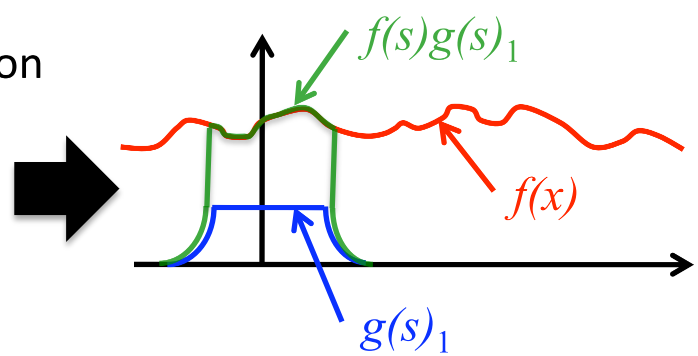
- recall: $f_k = \frac{1}{2\pi}\int f(x)e^{ikx}\,dx$

- a windowed Fourier transform can be expressed as:

$$f_{k,s} = \frac{1}{2\pi}\int f(s)g(s-x)e^{iks}\,ds$$

-where s is the position over a localized region

-the windowed transform is our convolution with a filter function in Fourier space



Wavelet decomposition (a brief overview)

- Wavelets offer an optimal space frequency decomposition, in 1D:

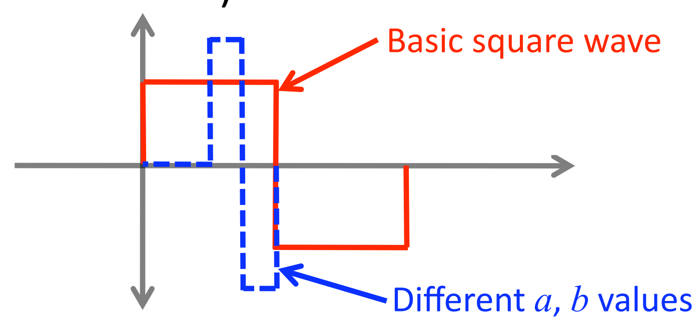
$$W_f(a,b) = |a|^{-\frac{1}{2}} \int f(x)\Psi\left(\frac{x-b}{a}\right) dx$$

where ***Ψ* is the basis function** (sometimes called the "Mother Wavelet"), ***b* translates** the basis function (location) and ***a* scales** the basis function (dilatation).

- A few different types of Wavelets exist, the main general classes (similar to Fourier) are:
  - -continuous transforms (what we wrote above)
  - -discrete transforms
    - Redundant discrete systems
    - Orthonormal systems (this is used in CVS)

- Example of a common wavelet (see listed references for others):
  - **Haar Wavelet**:

$$\Psi(x) = \begin{cases} 1 & 0 \le x < ½ \\ -1 & ½ \le x < 1 \end{cases}$$



Wavelet decomposition (a brief overview)

- How does wavelet decomposition break down a signal in space and time?