LES of Turbulent Flows: Lecture 6 (ME EN 7960-008)

Prof. Rob Stoll
Department of Mechanical Engineering
University of Utah

Spring 2011

Kolmogorov's Similarity hypothesis (1941)

Kolmogorov's 1st Hypothesis:

• smallest scales receive energy at a rate proportional to the dissipation of energy rate.

• motion of the very smallest scales in a flow depend only on:

a) rate of energy transfer from small scales: $\epsilon \left[\frac{L^2}{T^3}\right]$

b) kinematic viscosity: $\nu \left[\frac{L^2}{T}\right]$

With this he defined the Kolmogorov scales (dissipation scales):

- length scale: $\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$
- time scale: $\tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$
- velocity scale: $\upsilon = (\nu\epsilon)^{\frac{1}{4}}$

Re based on the Kolmolgorov scales => Re=1

Kolmogorov's Similarity hypothesis (1941)

From our scales we can also form the ratios of the largest to smallest scales in the flow (using $\ell_o,\ U_o,\ t_o$).
Note: dissipation at large scales => $\epsilon \sim \frac{U_o^3}{\ell_o}$

- length scale:

$$\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} \sim \left(\frac{\nu^3 \ell_o}{U_o^3}\right)^{\frac{1}{4}} \Rightarrow \frac{\eta}{\ell_o^{1/4}} \sim \frac{\nu^{3/4}}{U_o^{3/4}} \Rightarrow \boxed{\frac{\eta}{\ell_o} \sim \frac{\nu^{3/4}}{U_o^{3/4}\ell_o^{3/4}} \sim Re^{-3/4}}$$

- velocity scale:

$$v = (\nu\epsilon)^{\frac{1}{4}} \sim \left(\frac{\nu U_o^3}{\ell_o}\right)^{\frac{1}{4}} \Rightarrow \frac{v}{U_o^{3/4}} \sim \frac{\nu^{1/4}}{\ell_o^{1/4}} \Rightarrow \boxed{\frac{v}{U_o} \sim Re^{-1/4}}$$

- time scale:

$$\tau = \frac{\eta}{v} \Rightarrow \boxed{\frac{\tau}{t_o} \sim Re^{-1/2}}$$

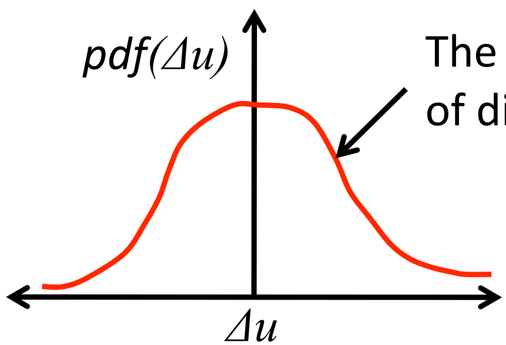
For very high-Re flows (e.g., Atmosphere) we have a range of scales that is small compared to ℓ_o but large compared to η. As *Re* goes up, η/ℓ_o goes down and we have a larger separation between large and small scales.

Kolmogorov's Similarity hypothesis (1941)

Kolmolgorov's 2nd Hypothesis:

In Turbulent flow, a range of scales exists at very high Re where statistics of motion in a range ℓ (for $\ell_o >> \ell >> \eta$) have a universal form that is determined only by ϵ (dissipation) and independent of ν (kinematic viscosity).

• Kolmogorov formed his hypothesis and examined it by looking at the pdf of velocity increments Δu.



The moments of this pdf are the structure functions of different order (e.g., 2nd, 3rd, 4th, etc.)

variance skewness kurtosis

- Another way to look at this (equivalent to structure functions) is to examine what it means for $E(k)$
- Recall $E(k)dk = \text{t.k.e. contained between } k \text{ and } k + dk$

Kolmogorov's Similarity Hypothesis (1941)

- What are the implications of Kolmolgorov's hypothesis for E(*k*)?

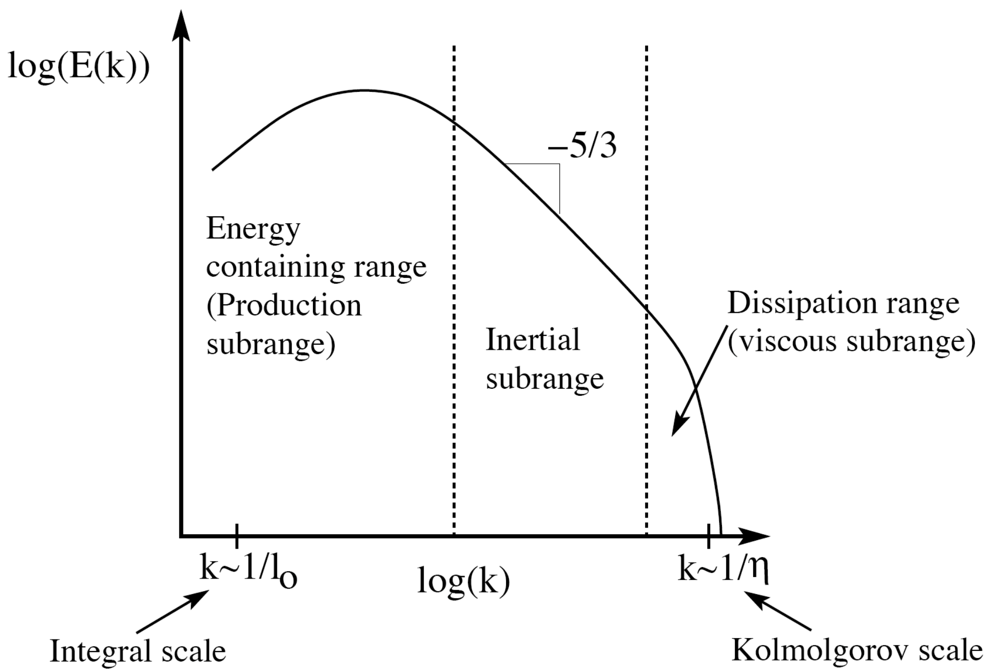
$$\mathrm{K41} \Rightarrow E(k) = f(k, \epsilon)$$

By dimensional analysis we can find that:

$$E(k) = c_k \epsilon^{2/3} k^{-5/3}$$

- This expression is valid for the range of length scales ℓ where $\ell_o >> \ell >> \eta$ and is usually called the inertial subrange of turbulence.

- graphically:



Degrees of freedom and numerical simulations

• We now have a description of turbulence and the range of energy containing scales (the dynamic range) in turbulence

• In CFD we need to discretize the equations of motion (see below) using either difference approximations (finite differences) or as a finite number of basis functions (e.g., Fourier transforms)

• To capture all the dynamics (degrees of freedom) of a turbulent flow we need to have a grid fine enough to capture the smallest **and** largest motions (η and ℓ_o)

• From K41 we know $\frac{\eta}{\ell_o} \sim Re^{-3/4}$ and we have a continuous range of scales between η and ℓ_o

• We need $\frac{\ell_o}{\eta} \sim Re^{3/4}$ in each direction. Turbulence is 3D => we need N~$Re^{9/4}$ points.

Degrees of freedom and numerical simulations

- When will we be able to directly simulate all the scales of motion in a turbulent flow? (Voller and Porté-Agel, 2002, see handouts for the full paper)

In the mid 1960s Gordon Moore, the co-founder of Intel, made the observation that computer power, P, measured by the number of transistors that could be fit onto a chip, doubled once every 1.5 years [1]. This law, which has performed extremely well over the proceeding 30 or so years, can be stated in mathematical terms as

$$P = A2^{0.6667Y}, \qquad (1)$$

where A is the computer power at the reference year $Y = 0$.

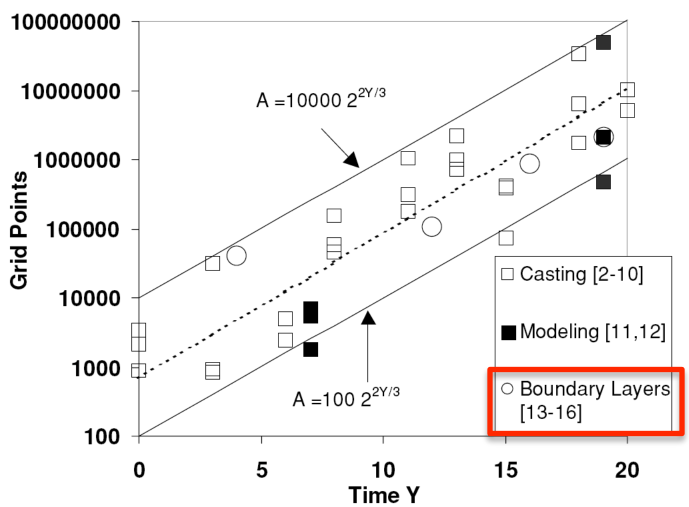


FIG. 1. Log of three largest grid sizes from each volume plotted against year.

TABLE II
Expected Year (±5) That the Given Direct Simulation Will Be Possible If Grid Size Increases Are Bound by Eq. (2)

Simulation	Domain length scale	Resolution length scale	Grid points required	Expected year (±5 years)
2-D casting	0.1 m	1 µm (dendrite tip)	10^{10}	2015
2-D casting	1 m	1 µm (dendrite tip)	10^{12}	2025
3-D casting	0.1 m	1 µm (dendrite tip)	10^{15}	2040
Boundary layer	100 m	1 mm	10^{15}	2040
2-D casting	0.1 m	1 nm (lattice space)	10^{16}	2045
3-D casting	1 m	1 µm (dendrite tip)	10^{18}	2055
2-D casting	1 m	1 nm (lattice space)	10^{18}	2055
Boundary layer	1 km	1 mm	10^{18}	2055
Boundary layer	10 km	1 mm	10^{21}	2070
3-D casting	0.1 m	1 nm (lattice space)	10^{24}	2085
3-D casting	1 m	1 nm (lattice space)	10^{27}	2100

Equations of Motion

- Turbulent flow (and fluid dynamics in general) can be mathematically described by the Navier-Stokes equations (see Bachelor, 1967 for a derivation of equations/Pope Ch 2)
- The primary goal of CFD (and LES) is to solve the discretized equations of motion.
- we use the continuum hypothesis (e.g., $\eta >>$ mean free path of molecules) so that

$$\Rightarrow \; u_i = u_i(x_j, t) \text{ and } \rho = \rho(x_j, t)$$

- **Conservation of Mass:**

$$\left.\frac{dm}{dt}\right)_{sys} = 0$$

Using Reynolds Transport Theorem (RTT, see any fluids textbook)

$$\Rightarrow \left.\frac{dm}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \;\; \Rightarrow \text{Integral form}$$

Using Gauss's theorem and shrinking the control volume to an infinitesimal size:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \;\; \Rightarrow \text{differential form}$$

Equations of Motion

<u>Conservation of Momentum:</u> (Newton's 2nd law)

$$\sum \vec{F} = \left.\frac{d\left(m\vec{V}\right)}{dt}\right)_{sys}$$ Using RTT ➔

$$\frac{\partial}{\partial t}\int_{CV} \rho\vec{V}d\forall + \int_{CS} \vec{V}\rho\vec{V}\,d\vec{A} = \underbrace{\int_{CS} \mathbf{T}\cdot\hat{n}\,d\vec{A}}_{\text{shear stress}} + \underbrace{\int_{CV} \rho\vec{b}d\forall}_{\text{body forces}} \Rightarrow \text{ integral form}$$

- The shear stress tensor depends on molecular processes. For a Newtonian fluid ➔

$$\mathbf{T} = -\left(P + \frac{2}{3}\mu\nabla\cdot\vec{V}\right)\mathbf{I} + 2\mu\mathbf{S}$$

Where $\mathbf{S} = \frac{1}{2}\left(\nabla\vec{V} + \nabla\vec{V}^T\right)$ or in index notation $S_{ij} = {}^1\!/_2\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the deformation (rate of strain) tensor and **I** is the unit tensor (or identity matrix)

- The equivalent index-notation (differential) form of the momentum equation is:

$$\frac{\partial\left(\rho u_i\right)}{\partial t} + \frac{\partial\left(\rho u_i u_j\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(2\mu S_{ij} - {}^2\!/_3\,\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right) - \frac{\partial P}{\partial x_i} + \rho g_i$$

where the stress has been split into shear (viscous) and normal (pressure) components.

Conservation of Energy

Conservation of Energy: (1st law of Thermodynamics)

For a system conservation of energy is: $\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys}$ or: **(in-out) + produced = stored**

- **(in-out)** is the convective flux of energy

-**Production** is the heat conducted in + the work done on the volume (e.g., thermal flux and shear stress)

• if we use $e = c_V T$ (specific internal energy)

• and define $q_i = -k\frac{\partial T}{\partial x_i}$ as the thermal conductive flux where c_V is the specific heat and T is temperature. We can derive the following differential form for energy

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}\left[u_i(P+E)\right] = \rho\dot{q} + \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left[u_j\left(2\mu S_{ij} - \tfrac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right)\right]$$

Where the total energy is: $E = e + \tfrac{1}{2}u_i u_i$