

LES of Turbulent Flows: Lecture 4 (ME EN 7960-008)

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Fourier Transforms

Fourier Transforms are a common tool in fluid dynamics (see Pope, Appendix D-G, Stull handouts online)

Some uses:

- Analysis of turbulent flow
- Numerical simulations of N-S equations
- Analysis of numerical schemes (modified wavenumbers)
- consider a periodic function $f(x)$ (could also be $f(t)$) on a domain of length 2π
- The Fourier representation of this function (or a general signal) is:

$$f(x) = \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx} \quad *$$

- where k is the wavenumber (frequency if $f(t)$)

- \hat{f}_k are the Fourier coefficients which in general are complex

Fourier Transforms

- why pick e^{ikx} ?

- answer: orthogonality
$$\int_0^{2\pi} e^{i(k-k')x} dx = \begin{cases} 0 & \text{if } k \neq k' \\ 2\pi & \text{if } k = k' \end{cases}$$

- a big advantage of orthogonality is independence between Fourier modes

- e^{ix} is independent of e^{i2x} just like we have with Cartesian coordinates where i, j, k are all independent of each other.

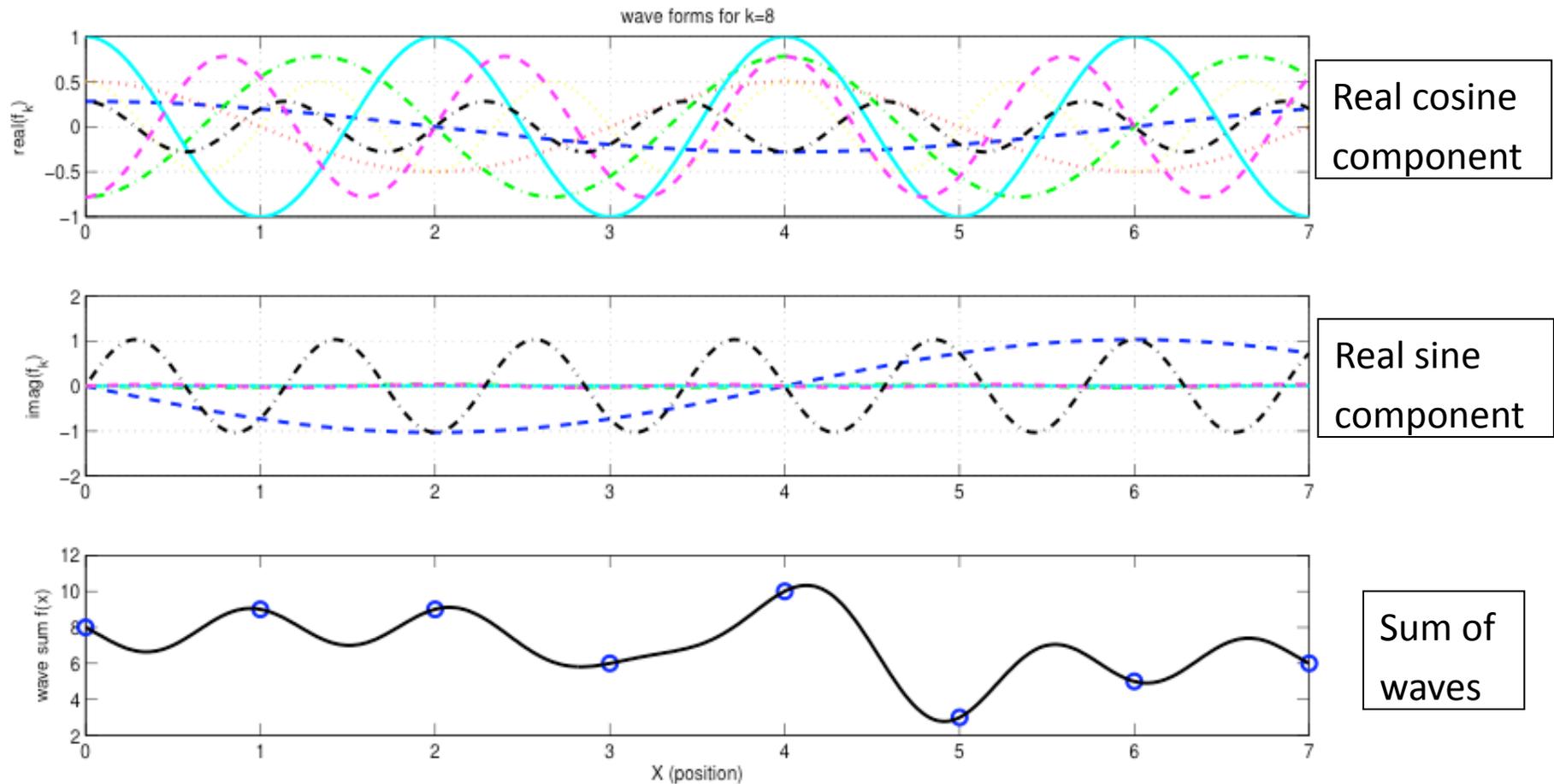
- what are we actually doing?

- recall from Euler's formula that $e^{ix} = \cos(x) - i \sin(x)$

- the Fourier transform decomposes a signal (space or time) into sine and cosine wave components of different amplitudes and wave numbers (or frequencies).

Fourier Transforms

- Fourier Transform example (from Stull, 88 see example: FourierTransDemo.m)



Fourier Transforms

- The Fourier representation given by \ast is a representation of a series as a function of sine and cosine waves. It takes $f(x)$ and transforms it into wave space
- Fourier Transform pair:** consider a periodic function on a domain of 2π

$$f_k = F\{f(x)\} \equiv \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-ikx} dx \rightarrow \text{forward transform}$$

$$f(x) = F\{\hat{f}_k\}^{-1} \equiv \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx} \rightarrow \text{backwards transform}$$

- The **forwards transform** moves us into Fourier (or wave) space and the **backwards transform** moves us from wave space back to real space.
- An alternative form of the Fourier transform \ast (using Euler's) is:

$$f(x) = a_0 + \sum_{k=1}^{k=\infty} a_k \cos(kx) - b_k \sin(kx)$$

Where a_k and b_k are now the real and imaginary components of f_k , respectively

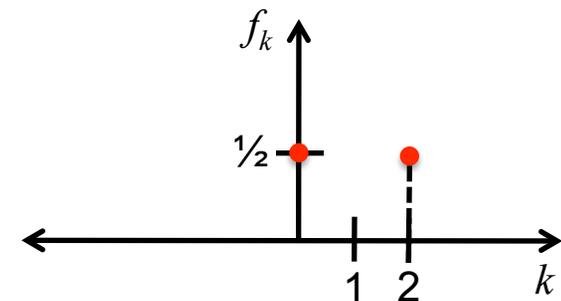
Locality in real and wave space

- It is important to note that something that is non local in physical (real) space can be very local in Fourier space and something that is local in physical space can be non local in Fourier space.

- Example 1:**
 $f(x) = \cos^2 x = \frac{1 + \cos 2x}{2}$ (a wave, very non local in physical space)

Fourier modes are :

$$a_0 = 1/2, a_2 = 1/2 \text{ all other } a_k, b_k = 0$$

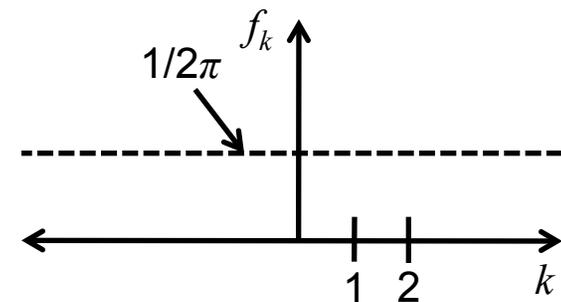


- Example 2:**
 $f(x) = \delta(x)$ the Dirac delta function (very non local in physical space)

$$\hat{f}_k = \frac{1}{2\pi} \int \delta(x) e^{-ikx} dx$$

recall by definition $\int \delta(x - a) f(x) dx = f(a)$

$$\Rightarrow \int = 1 \text{ for any value } k \Rightarrow \hat{f}_k = \frac{1}{2\pi}$$



Fourier Transform Properties

1. If $f(x)$ is real then:

$$\hat{f}_k = \hat{f}_k^* \quad (\text{Complex conjugate})$$

2. Parseval's Theorem:

$$\frac{1}{2\pi} \int_0^{2\pi} f(x) f^*(x) dx = \sum_{k=-\infty}^{k=\infty} \hat{f}_k \hat{f}_k^*$$

3. The Fourier representation is the best possible representation for $f(x)$ in the sense that the error:

$$e = \int_0^{2\pi} \left| f(x) - \sum_{k=-N}^N c_k e^{ikx} \right|^2 dx \quad \text{is minimum when } c_k = \hat{f}_k$$

Discrete Fourier Transform

- Consider the periodic function f_j on the domain $0 \leq x \leq L$



Periodicity implies that $f_0 = f_N$

- Discrete Fourier Representation:

$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{i\frac{2\pi}{L} k x_j} \Rightarrow \text{backwards (inverse) transform}$$

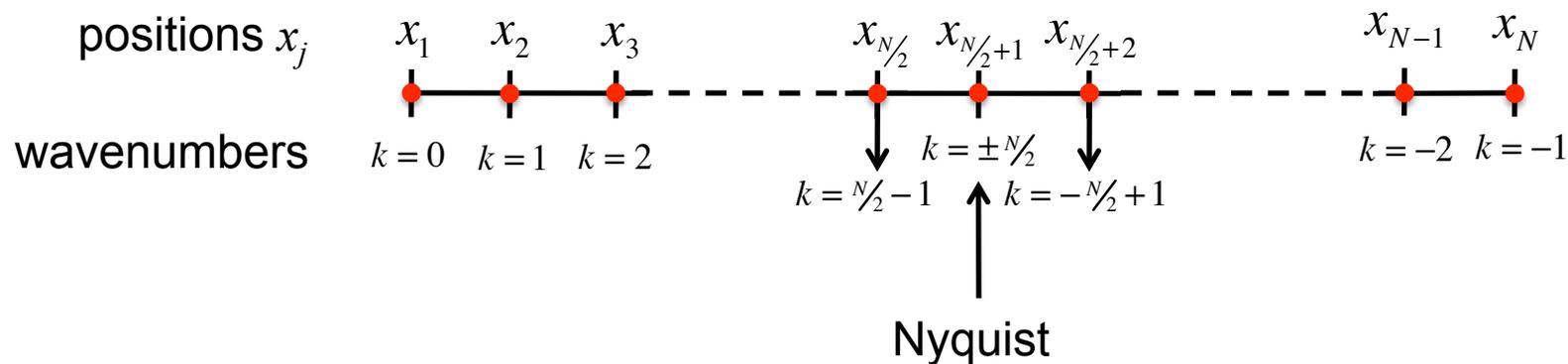
- we have:
 - known: f_j at N points
 - unknown: \hat{f}_k at k values (N of them)

- Using discrete orthogonality:

$$\hat{f}_k = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-i\frac{2\pi}{L} k x_j} \Rightarrow \text{forward transform}$$

Discrete Fourier Transform

- Discrete Fourier Transform (DFT) example and more explanation found in handouts section of website under Stull_DFT.pdf or Pope appendix F and in example: FourierTransDemo.m
- Implementation of DFT by brute force $\rightarrow \mathbf{O}(N^2)$ operations
- In practice we almost always use a Fast Fourier Transform (FFT) $\rightarrow \mathbf{O}(N \cdot \log_2 N)$
- Almost all FFT routines (e.g., Matlab, FFTW, Intel, Numerical Recipes, etc.) save their data with the following format:



Fourier Transform Applications

Autocorrelation:

- We can use the discrete Fourier Transform to speed up the autocorrelation calculation (or in general any cross-correlation with a lag).

-Discretely $R_{ff}(s_l) = \sum_{j=0}^{N-1} f(x_j) f(x_j + s_l)$ this is $\mathbf{O}(N^2)$ operations
 correlation with itself \rightarrow

-if we express R_{ff} as a Fourier series

$$R_{ff}(s_l) = \sum_k \hat{R}_{ff} e^{iks_l} \Rightarrow R_{ff}(0) = \sum_k \hat{R}_{ff} \text{ and we can show that } R_{ff}(0) = \sum_k N |\hat{f}_k|^2$$

magnitude of the
Fourier coefficients

-how can we interpret this??

-In physical space

$$R_{ff}(0) = \sum_{j=0}^{N-1} f_j^2 \text{ (i.e. the mean variance)} \Rightarrow \sum_{j=0}^{N-1} f_j^2 = \sum_{k=-N/2}^{N/2-1} N |\hat{f}_k|^2$$

total contribution
to the variance

energy spectral density

Fourier Transform Applications

Energy Spectrum: (power spectrum, energy spectral density)

- If we look at specific k values from our autocorrelation function we have:

$$E(k) = N \left| \hat{f}_k \right|^2$$

where $E(k)$ is the energy spectral density

The square of the Fourier coefficients is the contribution to the variance by fluctuations of scale k (wavenumber or equivalently frequency)

- Typically (when written as) $E(k)$ we mean the contribution to the turbulent kinetic energy (tke) = $\frac{1}{2}(u^2 + v^2 + w^2)$ and we would say that $E(k)$ is the contribution to tke for motions of the scale (or size) k . For a single velocity component in one direction we would write $E_{11}(k_1)$.
- This means that the energy spectrum and the autocorrelation function form a Fourier Transform pair (see Pope for details)

