LES of Turbulent Flows: Lecture 3 (ME EN 7960-008)

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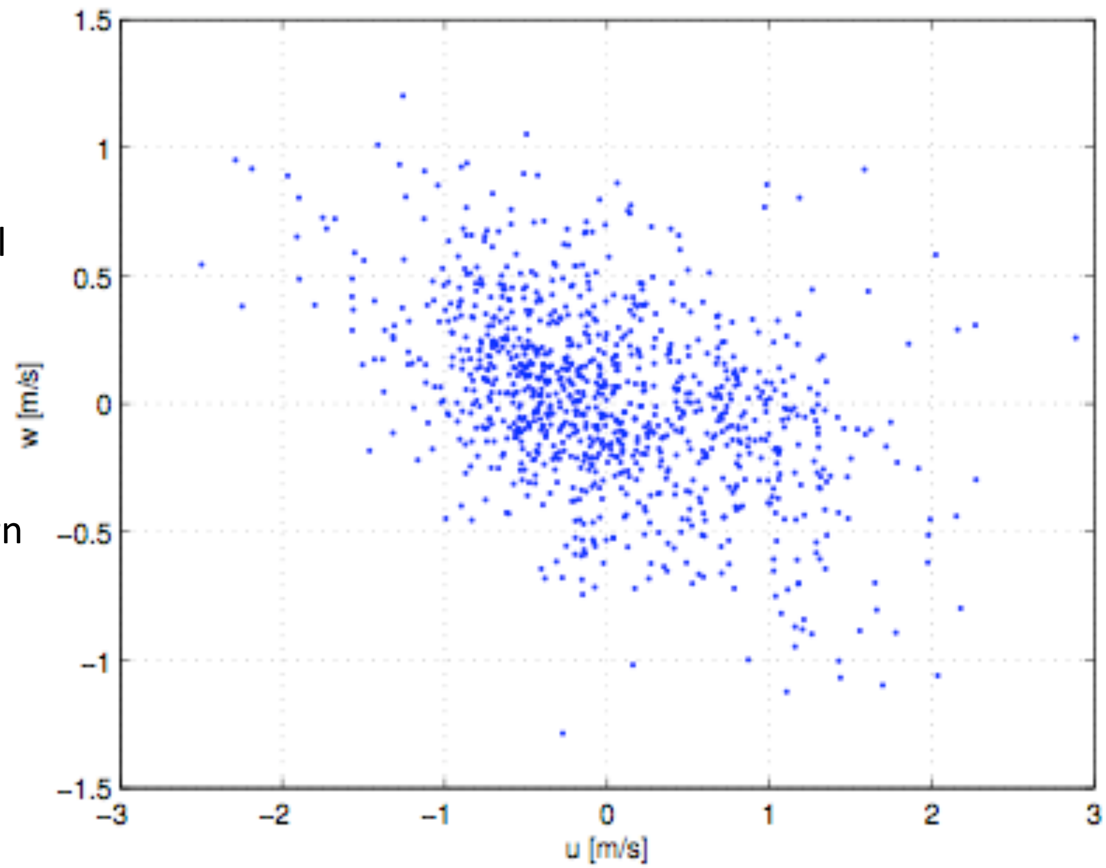
Spring 2011

Joint Random Variables

- So far the description has been limited to single Random variables but turbulence is governed by the Navier-Stokes equations which are a set of 3 coupled PDEs.
- We expect this will result in some correlation between different velocity components

- Example, turbulence data from the ABL: scatter plot of horizontal (u) and vertical (w) velocity fluctuations.

- The plot appears to have a pattern (negative slope)

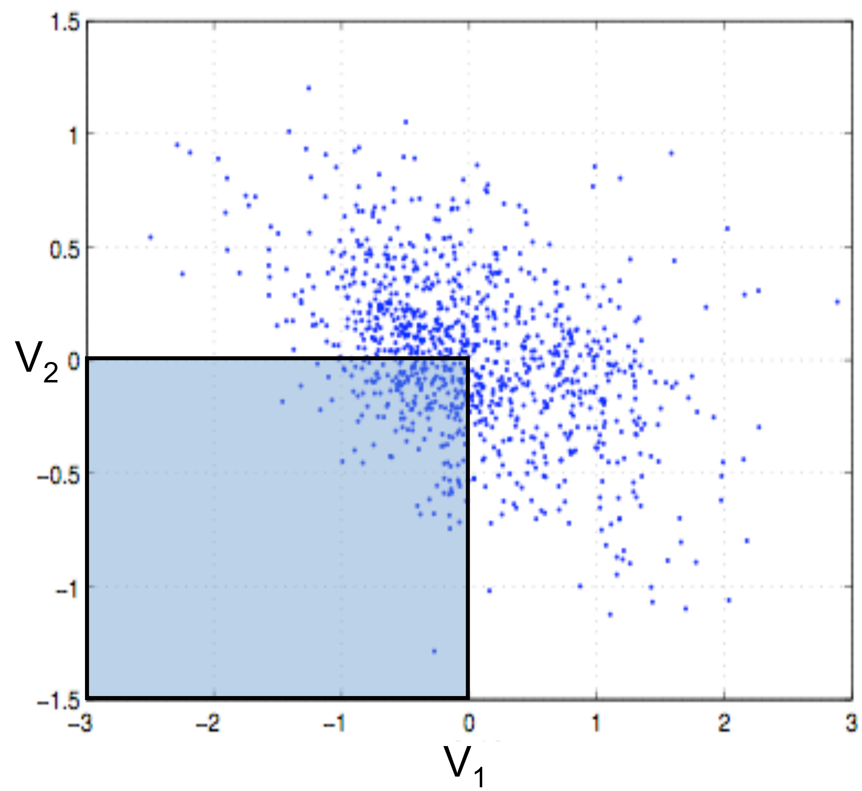


Joint Random Variables

- Joint Cumulative Density Function (joint CDF):

$$F_{12}(V_1, V_2) \equiv P\{U_1 < V_1, U_2 < V_2\}$$

Sample space of our random variables U_1 and U_2



- In the figure, the CDF is the probability that the variable (U_1 and U_2) lie within the shaded region
- The joint CDF has the following properties:
  - It is non decreasing
  - $F_{12}(-\infty, V_2) = P\{U_1 < -\infty, U_2 < V_2\} = 0$
  - $F_{12}(\infty, V_2) = P\{U_1 < \infty, U_2 < V_2\} =$

$P\{U_2 < V_2\} = F_2(V_2)$

i.e. since $U_1 < \infty$ is certain, the joing cdf $=$ the single variable cdf

Joint PDF

- The **joint PDF** is given by:

$$f_{12}(V_1,V_2)=\frac{\partial^2}{\partial V_1 \partial V_2}F_{12}(V_1,V_2)$$

- Similar to the single variable PDF, if we integrate over V_1 and V_2 we get the probability

$$P\{V_{1a}\le U_1<V_{1b},V_{2a}\le U_2<V_{2b}\}=\int_{V_{1a}}^{V_{1b}}\int_{V_{2a}}^{V_{2b}} f_{12}(V_1,V_2)dV_2 dV_1$$

- The joint PDF has the following properties:
  - $f_{12}(V_1,V_2)\ge 0$
  - $f_2(V_2)=\int_{-\infty}^{\infty} f_{12}(V_1,V_2) \Rightarrow$ the marginal PDF of U_2
  - $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{12}(V_1,V_2)dV_1 dV_2=1$
- Similar to a single variable, if we have Q(U_1, U_2) then

$$\langle Q(U_1,U_2)\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(V_1,V_2)f_{12}(V_1,V_2)dV_2 dV_1$$

- With this idea we can give a rigorous definition for a few important stats (next)

Important single point stats for joint variables

- **covariance**:

$$\mathrm{cov}(U_1,U_2)=\langle u_1u_2\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(V_1-\langle U_1\rangle\right)\left(V_2-\langle U_2\rangle\right)f_{12}(V_1,V_2)\,dV_2\,dV_1$$

- Or for discrete data

$$\mathrm{cov}(U_1,U_2)=\langle u_1u_2\rangle=\frac{1}{N-1}\sum_{j=1}^{N}\left(V_{1j}-\langle U_1\rangle\right)\left(V_{2j}-\langle U_2\rangle\right)$$

- We can also define the correlation coefficient (non dimensional)

$$\rho_{12}=\frac{\langle u_1u_2\rangle}{\left[\langle u_1^2\rangle\langle u_2^2\rangle\right]^{1/2}}$$

  - Note that $-1 \le \rho_{12} \le 1$ and negative value mean the variables are anti-correlated with positive values indicating a correlation

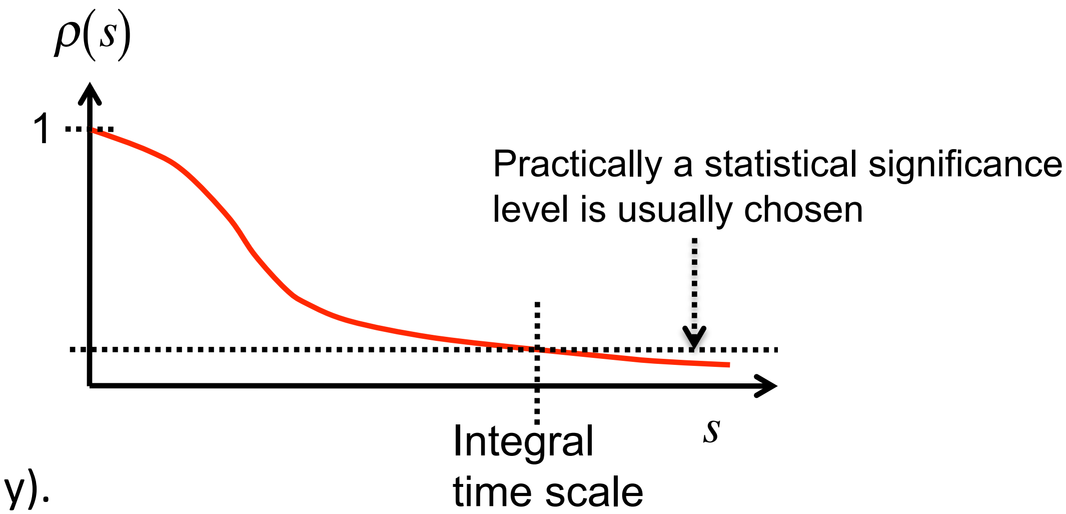
- **Practically speaking**, we find the PDF of a time (or space) series by:
  1. Create a histogram of the series (group values into bins)
  2. Normalize the bin weights by the total # of points

Two-point statistical measures

- **autocovariance:** measures how a variable changes (or the correlation) with different lags
$$R(s) \equiv \langle u(t)u(t+s) \rangle$$
- or the autocorrelation function
$$\rho(s) \equiv \langle u(t)u(t+s) \rangle / \langle u(t)^2 \rangle$$
  - These are very similar to the covariance and correlation coefficient
  - The difference is that we are now looking at the linear correlation of a signal with itself but at two different times (or spatial points), i.e. we lag the series.
- Discrete form of autocorrelation:
$$\rho(s_j) = \frac{\sum_{k=0}^{N-j-1} (u_k u_{k+j})}{\sum_{k=0}^{N-1} (u_k^2)}$$
- We could also look at the cross correlations in the same manner (between two different variables with a lag).
- Note that: $\rho(0) = 1$ and $|\rho(s)| \leq 1$

Two-point statistical measures

- In turbulent flows, we expect the correlation to diminish with increasing time (or distance) between points:



- We can use this to define an Integral time scale (or space). It is defined as the time lag where the integral $\int_0^\infty \rho(s)ds$ converges. and can be used to define the largest scales of motion (statistically).

- Another important 2 point statistic is the **structure function**:

$$D_n(r) \equiv \left\langle \left[U_1(x+r,t) - U_1(x,t)\right]^n \right\rangle$$

This gives us the average difference between two points separated by a distance r raised to a power n. In some sense it is a measure of the moments of the velocity increment PDF. Note the difference between this and the **autocorrelation which is statistical linear correlation** (ie multiplication) of the two points.