\textbf{Eddy viscosity Models}

- **Eddy Viscosity Models:** (Guerts pg 225; Pope pg 587)
  
  - In Lecture 8 we said an eddy-viscosity models are of the form:
    
    \begin{align*}
    \tau_{ij} &= -2\nu_T \tilde{S}_{ij} & \text{filtered strain rate} \\
    q_i &= -D_T \frac{\partial \tilde{\theta}}{\partial x_i} & \text{SGS Prandtl number} \\
    \end{align*}

    eddy-viscosity  
    eddy-diffusivity

  - This is the LES equivalent to 1\textsuperscript{st} order RANS closure (k-theory or gradient transport theory) and is an analogy to molecular viscosity (see Pope Ch. 10 for a review).
  - Turbulent fluxes are assumed to be proportional to the local velocity or scalar gradients.
  - In LES this is the assumption that stress is proportional to strain: $\tau_{ij} \sim \tilde{S}_{ij}$
  - The SGS eddy-viscosity $\nu_T$ must still be parameterized.
Eddy viscosity Models

- Dimensionally =>

\[ \nu_T = \left[ \frac{L^2}{T} \right] \]

- In almost all SGS eddy-viscosity models \( \nu_T \sim u^* l^* \)

- Different models use different \( u^* \) and \( l^* \)

- Recall from last time, we can interpret the eddy-viscosity as adding to the molecular viscosity so that the (dimensional) viscous term is:

\[ \frac{\partial}{\partial x_j} \left[ (\nu_T + \nu) \tilde{S}_{ij} \right] \]

**What does the model do?** We can see it effectively lowers the Reynolds number of the flow and for high Re (when \( 1/\text{Re} \geq 0 \)), it provides all of the energy dissipation.

Most LES models use a nonlinear eddy viscosity. What happens if we use a constant?

- We effectively run the simulation at a different (lower) Re
- This has implications for DNS. If we try to use DNS at a lower Re to examine phenomena that happens at a higher Re, unless our low-Re is high enough that Re-invariance assumptions apply we can make the analogy between our DNS and an LES with a SGS model that doesn’t properly reproduce the flow physics.
Smagorinsky Model

**Smagorinsky model**: (Smagorinsky, MWR 1963)
- One of the 1st and still most popular $\nu_T$ models for LES
- Originally developed for general circulation models (large-scale atmospheric), the model did not remove enough energy in this context.
- Applied by Deardorff (JFM, 1970) in the 1st LES.
- Uses Prandtl’s mixing length idea (1925) applied at the SGSs (see Pope Ch. 10 or Stull, 1988 for a full review of mixing length):

**In Prandtl’s mixing length**, for a general scalar quantity $q$ with an assumed linear profile:

- A turbulent eddy moves a parcel of air by an amount $z'$ towards a level $z$ where we have no mixing or other change
- $q'$ will differ from the surrounding by: $q' = -\left(\frac{\partial \langle q \rangle}{\partial z}\right) z'$
- ie it will change proportional to its local gradient
- Similarly, if velocity also has a linear profile:
  $$u' = -\left(\frac{\partial \langle u \rangle}{\partial z}\right) z'$$
Smagorinsky Model

-To move up a distance $z'$ our air parcel must have some vertical velocity $w'$

-If turbulence is such that $w' \sim u'$ then $w' = Cu'$ and we have 2 cases:
  \[
  \frac{\partial u}{\partial z} > 0 \Rightarrow w' = -Cu'
  \]
  \[
  \frac{\partial u}{\partial z} < 0 \Rightarrow w' = Cu'
  \]

-Combining these we get that:
  \[
  w' = C \left| \frac{\partial \langle u \rangle}{\partial z} \right| z'
  \]

-We $q'$ and $w'$ and now we can form a kinematic flux (conc. * velocity) by multiplying the two together:
  \[
  \langle q'w' \rangle = -C \langle (z')^2 \rangle \left| \frac{\partial \langle u \rangle}{\partial z} \right| \frac{\partial \langle q \rangle}{\partial z}
  \]

-Where $(z')^2$ is the variance a parcel moves and $C(z')^2$ is defined as the mixing length →
  \[
  \langle q'w' \rangle = -\ell^2 \left| \frac{\partial \langle u \rangle}{\partial z} \right| \frac{\partial \langle q \rangle}{\partial z}
  \]

-we can replace $q'$ with any variable in this relationship (e.g. $u'$)
Smagorinsky Model

• **Back to Smagorinsky model:**
  - Use Prandtl’s mixing length applied at the SGSs:
    \[ \nu_T = (C_S \Delta)^2 |\tilde{S}| \]
    length scale velocity scale
  - Where \( \Delta \) is the grid scale taken as \( \Delta = (\Delta_x \Delta_y \Delta_z)^{1/3} \) (Deardorff, 1970 or see Scotti et al., PofF 1993 for a more general description).
  - \(|\tilde{S}| = \sqrt{2\tilde{S}_{ij}\tilde{S}_{ij}}\) is the magnitude of the filtered strain rate tensor with units \([1/T]\) and serves as the velocity scale (think \( \partial(u) / \partial z \) in Prandtl’s theory) and \( C_S \Delta \) is our length scale (squared for dimensional consistency).
  - The final model is:
    \[ \tau_{ij} - \frac{1}{3} \tau_{kk}\tilde{\delta}_{ij} = -2 (C_S \Delta)^2 |\tilde{S}|\tilde{S}_{ij} \]
  - To close the model we need a value of \( C_S \) (usually called the Smagorinsky or Smagorinsky-Lilly coefficient)
Lilly’s Determination of $C_S$

- Lilly proposed a method to determine $C_S$ (IBM Symposium, 1967, see Pope page 587)
- Assume we have a high-Re flow $\Rightarrow \Delta$ can be taken to be in the inertial subrange of turbulence.
- The mean energy transfer across $\Delta$ must be balanced by viscous dissipation, on average (note for $\Delta$ in the inertial subrange this is not an assumption).
  
  \[
  \epsilon = \langle \Pi \rangle \quad \text{recall:} \quad \Pi = -\tau_{ij} \tilde{S}_{ij}
  \]
  
  - Using an eddy-viscosity model $\nu_T \Rightarrow \Pi = 2\nu_T \tilde{S}_{ij} \tilde{S}_{ij} = \nu_T |\tilde{S}|^2$
  - If we use the Smagorinsky model: $\nu_T = (C_S \Delta)^2 |\tilde{S}|$
  
  \[
  \Rightarrow \quad \Pi = (C_S \Delta)^2 |\tilde{S}|^3
  \]
  
  - The square of $|\tilde{S}|$ can be written as (see Pope pg 579 for details):
    
    \[
    |\tilde{S}|^2 = 2 \int_{0}^{\infty} k^2 \tilde{G}(k)^2 E(k) dk
    \]
  
  - Recall, for a Kolmogorov spectrum in the inertial subrange $E(k) \sim C_k \epsilon^{2/3} k^{-5/3}$
  - We can use this in our integral to obtain (see Pope pg. 579): $|\tilde{S}|^2 \approx a_f C_k \epsilon^{2/3} \Delta^{-4/3}$
Lilly’s Determination of $C_S$

• We can rearrange $|\tilde{S}|^2 \approx a_f C_k \varepsilon^{2/3} \Delta^{-4/3}$ to get:  
  $$\varepsilon = \left[ \frac{\langle |\tilde{S}|^2 \rangle}{a_f C_k \Delta^{-4/3}} \right]^{3/2}$$  

• Equating viscous dissipation and the average Smagorinsky SGS dissipation ($\varepsilon = \langle \Pi \rangle$):  
  $$\varepsilon = \langle (C_S \Delta)^2 |\tilde{S}|^3 \rangle$$

• if we now combine this equation with (*) above and do some algebra...  
  $$C_S = \frac{1}{(C_k a_f)^{3/4} \left( \frac{\langle |\tilde{S}|^3 \rangle}{\langle |\tilde{S}|^2 \rangle^{3/2}} \right)^{-1/2}}$$

• we can use the approximation $\langle |\tilde{S}|^3 \rangle \approx \langle |\tilde{S}|^2 \rangle^{3/2}$ and $a_f$ for a cutoff filter (see Pope)  
  $$\Rightarrow C_S = \frac{1}{\pi} \left( \frac{2}{3C_k} \right)^{3/4}$$

• $C_k$ is the Kolmogorov constant ($C_k \approx 1.5-1.6$) and with this value we get:  
  $$C_S \approx 0.17$$