LES of Turbulent Flows: Lecture 5
(ME EN 7960-003)

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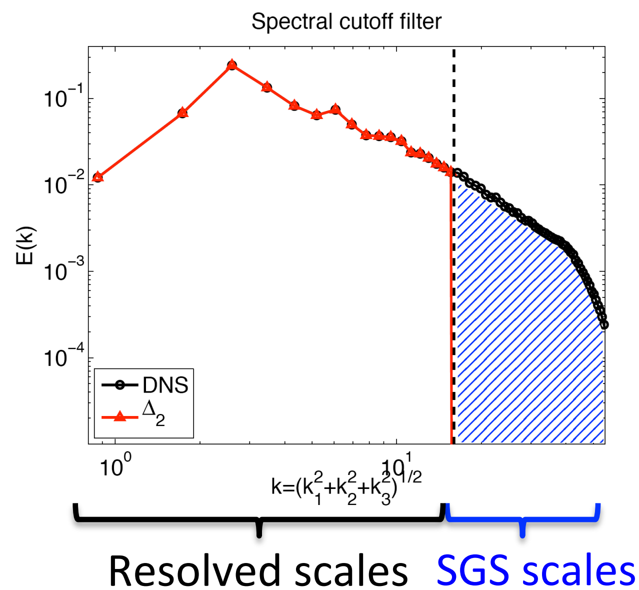
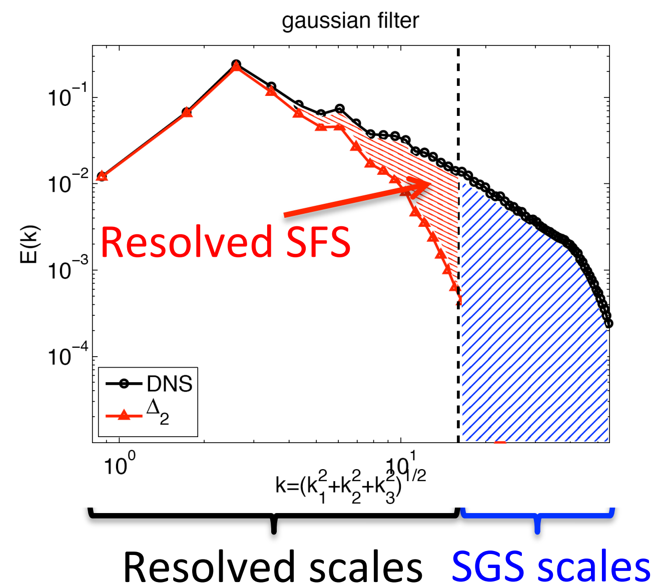
Decomposition of Turbulence for real filters

The LES filter can be used to decompose the velocity field into resolved and subfilter scale (SFS) components

$$\phi(\vec{x},t) = \widetilde{\phi}(\vec{x},t) + \phi'(\vec{x},t)$$

We can use our filtered DNS fields to look at how the choice of our filter kernel affects this separation in wavespace





The Gaussian filter (or box filter) does not have as compact of support in wavespace as the cutoff filter. This results in attenuation of energy at scales larger than the filter scale. The scales affected by this attenuation are referred to as **Resolved SFSs**.

Equations of Motion

- We want to apply our filters to the N-S equations, **for incompressible flow** (lecture 3):

-**Conservation of Mass:**

$$\frac{\partial u_i}{\partial x_i} = 0$$

-**Conservation of Momentum**:

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{\mathrm{Re}} \frac{\partial^2 u_i}{\partial x_j^2} + F_i$$

-**Conservation of Scalar**:

$$\frac{\partial \theta}{\partial t} + \frac{\partial u_j \theta}{\partial x_j} = -\frac{1}{Sc\,\mathrm{Re}} \frac{\partial^2 \theta}{\partial x_j^2} + Q$$

Filtering the incompressible N-S equations

- What happens when we apply one of the above filters to the N-S equations?

-**Conservation of Mass:**

filtering both sides of the conservation of mass: $\widetilde{\frac{\partial u_i}{\partial x_i}} = 0 \Rightarrow \frac{\partial \tilde{u}_i}{\partial x_i} = 0$

where we have used the property of LES filters => $\widetilde{\frac{\partial \phi}{\partial \vec{x}}} = \frac{\partial \tilde{\phi}}{\partial \vec{x}}$ and (~) denotes the filtering operation.

-Conservation of Momentum:

Using the filter properties $\tilde{\tilde{a}} = a$, $\widetilde{\phi + \zeta} = \tilde{\phi} + \tilde{\zeta}$ and $\widetilde{\frac{\partial \phi}{\partial \vec{x}}} = \frac{\partial \tilde{\phi}}{\partial \vec{x}}$ we can write the momentum equation as:

$$\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \widetilde{u_i u_j}}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} + F_i$$

The 2nd term on the LHS (convective term) now contains the unknown $\widetilde{u_i u_j}$ we can rewrite this term to obtain the standard LES equations for incompressible flow

Filtering the incompressible N-S equations

We can add and subtract $\tilde{u}_i\tilde{u}_j$ from the convective term:

$$\frac{\partial \widetilde{u_i u_j}}{\partial x_j} = \frac{\partial\left(\widetilde{u_i u_j} + \tilde{u}_i\tilde{u}_j - \tilde{u}_i\tilde{u}_j\right)}{\partial x_j} = \frac{\partial \tilde{u}_i\tilde{u}_j}{\partial x_j} + \frac{\partial\left(\widetilde{u_i u_j} - \tilde{u}_i\tilde{u}_j\right)}{\partial x_j}$$

Putting this back in the momentum equation and rearranging we have

$$\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \tilde{u}_i\tilde{u}_j}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} + F_i$$

SFS force vector

where $\boxed{\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i\tilde{u}_j}$ is the **subfilter scale (SFS) stress tensor**

- For the **scalar concentration** equation we can go through a similar process to obtain:

$$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial \tilde{u}_i\tilde{\theta}}{\partial x_i} = \frac{1}{Sc\,\mathrm{Re}}\frac{\partial^2 \tilde{\theta}}{\partial x_i^2} - \frac{\partial q_i}{\partial x_i} + Q$$

Where $q_j = \widetilde{u_j\theta} - \tilde{u}_j\tilde{\theta}$ is the **SFS flux**

LES filtered Equations for incompressible flow

•Mass: $$\frac{\partial \tilde{u}_i}{\partial x_i} = 0$$ ⓐ

•Momentum: $$\frac{\partial \tilde{u}_i}{\partial t} + \frac{\partial \tilde{u}_i \tilde{u}_j}{\partial x_j} = -\frac{\partial \tilde{p}}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \frac{\partial \tau_{ij}}{\partial x_j} + F_i$$ ⓑ

•Scalar: $$\frac{\partial \tilde{\theta}}{\partial t} + \frac{\partial \tilde{u}_i \tilde{\theta}}{\partial x_i} = \frac{1}{Sc\,\mathrm{Re}}\frac{\partial^2 \tilde{\theta}}{\partial x_i^2} - \frac{\partial q_i}{\partial x_i} + Q$$

•SFS stress: $$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j$$

•SFS flux: $$q_j = \widetilde{u_j \theta} - \tilde{u}_j \tilde{\theta}$$

• we've talked about variance (or energy) when discussing turbulence and filtering

• when we examined application of the LES filter at scale Δ we looked at the effect of the filter on the distribution of energy with scale.

• A natural way to extend our examination of scale separation and energy is to look at the evolution of the filtered variance or kinetic energy

The filtered kinetic energy equation

- **filtered kinetic energy equation** for incompressible flow

  -We can define the total filtered kinetic energy by: $\tilde{E} = \frac{1}{2}\widetilde{u_i u_i}$

  -We can decompose this in the standard way by:

$$\tilde{E} = \underbrace{\tilde{E}_f}_{\text{Resolved Kinetic energy}} + \underbrace{k_r}_{\text{SFS Kinetic energy}}$$

  -The SFS kinetic energy (or residual kinetic energy) can be defined as:

$$k_r = \tfrac{1}{2}\left(\widetilde{u_i u_i} - \tilde{u}_i\tilde{u}_i\right)$$

(see Pope pg. 585 or Piomelli et al., Phys Fluids A, 1991)

  -The resolved (filtered) kinetic energy is then given by:

$$\tilde{E}_f = \tfrac{1}{2}\tilde{u}_i\tilde{u}_i$$

The filtered kinetic energy equation

- We can develop an equation for $\tilde{E}_f$ by multiplying equation b on page 2 by $\tilde{u}_i$:

$$\boxed{\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t}} + \boxed{\tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j}} = -\boxed{\tilde{u}_i \frac{1}{\rho}\frac{\partial \tilde{P}}{\partial x_i}} + \nu \tilde{u}_i \frac{\partial^2 \tilde{u}_i}{\partial x_j^2} - \tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j}$$

- Applying the product rule to the terms in the squares:

$$\frac{\partial \tilde{u}_i \tilde{u}_i}{\partial t} = \tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t} \Rightarrow \boxed{\tilde{u}_i \frac{\partial \tilde{u}_i}{\partial t} = \tfrac{1}{2} \frac{\partial \tilde{u}_i \tilde{u}_i}{\partial t}}$$

$$\tilde{u}_j \frac{\partial \tilde{u}_i \tilde{u}_i}{\partial x_j} = \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} \Rightarrow \boxed{\tilde{u}_i \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \tfrac{1}{2} \tilde{u}_j \frac{\partial \tilde{u}_i \tilde{u}_i}{\partial x_j}}$$

$$\frac{\partial \tilde{P} \tilde{u}_i}{\partial x_i} = \tilde{P} \cancelto{0 \text{ (eqn a)}}{\frac{\partial \tilde{u}_i}{\partial x_i}} + \tilde{u}_i \frac{\partial \tilde{P}}{\partial x_i} \Rightarrow \boxed{\tilde{u}_i \frac{\partial \tilde{P}}{\partial x_i} = \frac{\partial \tilde{P} \tilde{u}_i}{\partial x_i}}$$

- Using our definition of $\tilde{E}_f$:

$$\frac{\partial \tilde{E}_f}{\partial t} + \tilde{u}_j \frac{\partial \tilde{E}_f}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{u}_i \tilde{P}}{\partial x_i} + \underbrace{\nu \tilde{u}_i \frac{\partial^2 \tilde{u}_i}{\partial x_j^2}}_{\triangle} - \underbrace{\tilde{u}_i \frac{\partial \tau_{ij}}{\partial x_j}}_{\triangledown}$$

The filtered kinetic energy equation

- term △ :

$$\frac{\partial^2 \tilde{u}_i\tilde{u}_i}{\partial x_j^2}=\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_j}\tilde{u}_i\tilde{u}_i\right]\xrightarrow{\text{Product rule}}\boxed{=\frac{\partial}{\partial x_j}\left[2\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j}\right]\xrightarrow{\text{Product rule}}=2\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j}+\underbrace{2\tilde{u}_i\frac{\partial^2 \tilde{u}_i}{\partial x_j^2}}}$$

Looks just like △ (without ν)

- using squared equation and divide by 2 and multiplying by ν:

$$\nu\tilde{u}_i\frac{\partial^2 \tilde{u}_i}{\partial x_j^2}=\underbrace{\nu\frac{\partial}{\partial x_j}\left[\tilde{u}_i\frac{\partial \tilde{u}_i}{\partial x_j}\right]}-\underbrace{\nu\frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_i}{\partial x_j}} \qquad \text{recall that } \tilde{S}_{ij}=\frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j}+\frac{\partial \tilde{u}_j}{\partial x_i}\right)$$

Uses symmetry of $\tilde{S}_{ij}$ and tensor contraction

$$\rightarrow 2\nu\frac{\partial}{\partial x_j}\left[\tilde{u}_i\tilde{S}_{ij}\right]-\underbrace{2\nu\tilde{S}_{ij}\tilde{S}_{ij}}_{\varepsilon_f}$$

- term ▽ :

$$\frac{\partial \tilde{u}_i\tau_{ij}}{\partial x_j}=\tilde{u}_i\frac{\partial \tau_{ij}}{\partial x_j}+\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j}\Rightarrow\tilde{u}_i\frac{\partial \tau_{ij}}{\partial x_j}=\frac{\partial \tilde{u}_i\tau_{ij}}{\partial x_j}-\underbrace{\tau_{ij}\frac{\partial \tilde{u}_i}{\partial x_j}}_{\tau_{ij}\tilde{S}_{ij}=\Pi}$$

- Combining everything back together:

$$\frac{\partial \tilde{E}_f}{\partial t}+\tilde{u}_j\frac{\partial \tilde{E}_f}{\partial x_j}=-\frac{1}{\rho}\frac{\partial \tilde{u}_i\tilde{p}}{\partial x_j}-\frac{\partial \tilde{u}_i\tau_{ij}}{\partial x_j}-2\nu\frac{\partial \tilde{u}_i\tilde{S}_{ij}}{\partial x_j}-\varepsilon_f-\Pi$$

"storage" of $\tilde{E}_f$ | advection of $\tilde{E}_f$ | pressure transport | transport of SFS stress τ_{ij} | transport of viscous stress | dissipation by viscous stress | SFS dissipation

Transfer of energy between resolved and SFSs

• The **SFS dissipation** Π in the resolved kinetic energy equation is a sink of resolved kinetic energy (it is a source in the k_r equation) and represents the transfer of energy from resolved SFSs. It is equal to:

$$\boxed{\Pi = -\tau_{ij}\tilde{S}_{ij}}$$

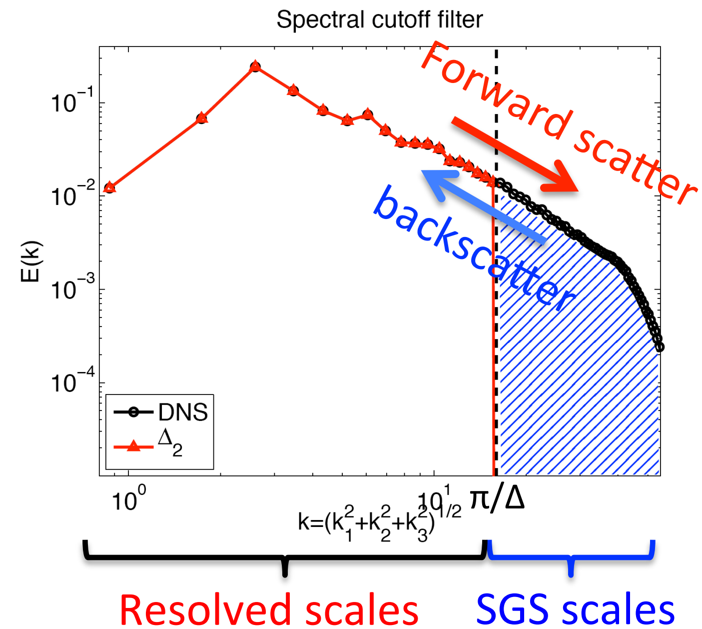
• It is referred to as the SFS dissipation as an analogy to viscous dissipation (and in the inertial subrange Π = viscous dissipation).

• On average Π **drains energy** (transfers energy down to smaller scale) from the resolved scales.

• Instantaneously (locally) Π can be positive **or** negative.

-When Π is negative (transfer from SFS➔Resolved scales) it is typically termed **backscatter**

-When Π is positive it is sometimes referred to as **forward scatter**.



Transfer of energy between resolved and SFSs

- Its informative to compare our resolved kinetic energy equation to the mean kinetic energy equation (derived in a similar manner, see Pope pg. 124; Stull 1988 ch. 5)

$$\frac{\partial\langle E\rangle}{\partial t}+\langle u_i\rangle\frac{\partial\langle E\rangle}{\partial x_j}+\frac{1}{\rho}\frac{\partial\langle u_i\rangle\langle P\rangle}{\partial x_j}-\frac{\partial}{\partial x_j}2\nu\langle u_i\rangle\langle S_{ij}\rangle=-P-\langle\varepsilon\rangle$$

shear production $=\langle u_i'u_j'\rangle\frac{\partial\langle u_i\rangle}{\partial x_j}$

mean dissipation $=2\nu\langle S_{ij}\rangle\langle S_{ij}\rangle$

- For high-Re flow, with our filter in the inertial subrange:

$$\langle\tilde{E}_f\rangle=\langle E\rangle$$

- The dominant sink for $\langle\tilde{E}_f\rangle$ is Π while for $\langle E\rangle$ it is $\langle\varepsilon\rangle$ (rate of dissipation of energy). For high-Re flow we therefore have:

$$\langle\Pi\rangle\approx\langle\varepsilon\rangle$$

- Recall from K41, $\langle\varepsilon\rangle$ is proportional to the transfer of energy in the inertial subrange
  ➔ Π will have a strong impact on energy transfer and the shape of the energy spectrum in LES.

- Calculating the correct average Π is another necessary (but not sufficient) condition for an LES SFS model (to go with our N-S invariance properties from Lecture 7).