LES of Turbulent Flows: Lecture 3
(ME EN 7960-003)

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Turbulent Flow Properties

- Review from Previous

Properties of Turbulent Flows:

1. Unsteadiness: u=f(**x**,t)
2. 3D: **x**=f(x_i) for any turbulent flow
3. High vorticity: $\omega = \vec{\nabla} \times \vec{u}$ or $\omega_k = \epsilon_{ijk} \frac{\partial}{\partial x_i} u_j \hat{e}_k$
4. Mixing effect: gradients are reduced by turbulence
5. A continuous spectrum (range) of scales: energy cascade described broadly by Kolmolgorov's 1st and 2nd hypothesis

Kolmogorov's Similarity hypothesis (1941)

Kolmogorov's 1st Hypothesis:

• smallest scales receive energy at a rate proportional to the dissipation of energy rate.

With this he defined the Kolmogorov scales (dissipation scales):

- length scale: $\eta = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}}$
- time scale: $\tau = \left(\frac{\nu}{\epsilon}\right)^{\frac{1}{2}}$
- velocity scale: $v = (\nu\epsilon)^{\frac{1}{4}}$

From our scales we can also form the ratios of the largest to smallest scales in the flow (using $\ell_o,\ U_o,\ t_o$).

- length scale: $\frac{\eta}{\ell_o} \sim Re^{-3/4}$
- velocity scale: $\frac{v}{U_o} \sim Re^{-1/4}$
- time scale: $\frac{\tau}{t_o} \sim Re^{-1/2}$

Kolmogorov's Similarity hypothesis (1941)

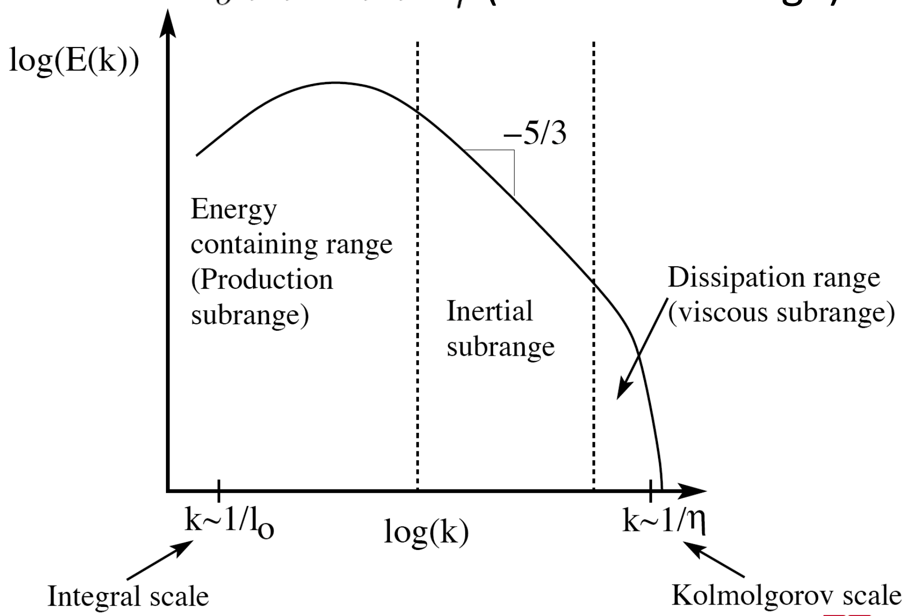
Kolmolgorov's 2nd Hypothesis:

A range of scales exists at very high Re where statistics of motion in a range $\ell_o >> \ell >> \eta$ have a universal form that is determined only by ϵ and independent of ν .

- We can examine this through: $E(k)dk = \text{t.k.e. contained between } k \text{ and } k + dk$
- What are the implications of Kolmolgorov's hypothesis for E(*k*)? $\text{K41} \Rightarrow E(k) = f(k, \epsilon)$

By dimensional analysis we can find that: $E(k) = c_k \epsilon^{2/3} k^{-5/3}$

- This expression is valid for $\ell_o >> \ell >> \eta$ (inertial subrange)



Degrees of freedom and numerical simulations

- We now have a description of turbulence and the range of energy containing scales (the dynamic range) in turbulence

- In CFD we need to discretize the equations of motion (see below) using either difference approximations (finite differences) or as a finite number of basis functions (e.g., Fourier transforms)

- To capture all the dynamics (degrees of freedom) of a turbulent flow we need to have a grid fine enough to capture the smallest **and** largest motions (η and ℓ_o)

- From K41 we know $\frac{\eta}{\ell_o} \sim Re^{-3/4}$ and we have a continuous range of scales between η and ℓ_o

- We need $\frac{\ell_o}{\eta} \sim Re^{3/4}$ in each direction. Turbulence is 3D => we need $N \sim Re^{9/4}$ points.

Degrees of freedom and numerical simulations

- When will we be able to directly simulate all the scales of motion in a turbulent flow? (Voller and Porté-Agel, 2002, see handouts for the full paper)

In the mid 1960s Gordon Moore, the co-founder of Intel, made the observation that computer power, P, measured by the number of transistors that could be fit onto a chip, doubled once every 1.5 years [1]. This law, which has performed extremely well over the proceeding 30 or so years, can be stated in mathematical terms as

$$P = A2^{0.6667Y}, \qquad (1)$$

where A is the computer power at the reference year $Y = 0$.

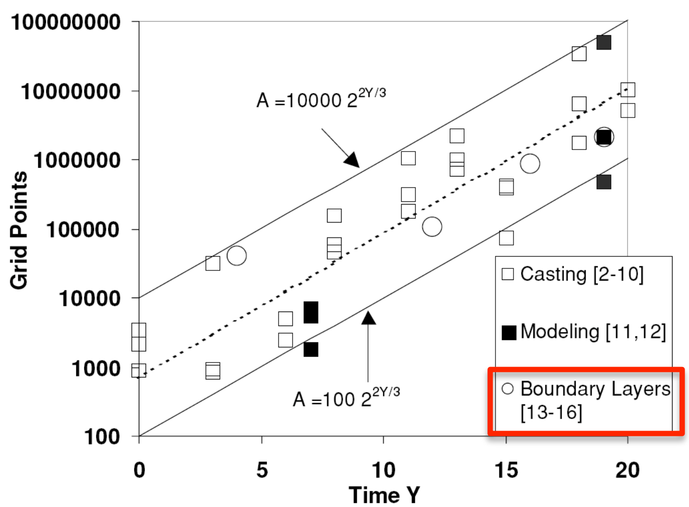


FIG. 1. Log of three largest grid sizes from each volume plotted against year.

TABLE II
Expected Year (±5) That the Given Direct Simulation Will Be Possible If Grid Size Increases Are Bound by Eq. (2)

Simulation	Domain length scale	Resolution length scale	Grid points required	Expected year (±5 years)
2-D casting	0.1 m	1 μm (dendrite tip)	10^{10}	2015
2-D casting	1 m	1 μm (dendrite tip)	10^{12}	2025
3-D casting	0.1 m	1 μm (dendrite tip)	10^{15}	2040
Boundary layer	100 m	1 mm	10^{15}	2040
2-D casting	0.1 m	1 nm (lattice space)	10^{16}	2045
3-D casting	1 m	1 μm (dendrite tip)	10^{18}	2055
2-D casting	1 m	1 nm (lattice space)	10^{18}	2055
Boundary layer	1 km	1 mm	10^{18}	2055
Boundary layer	10 km	1 mm	10^{21}	2070
3-D casting	0.1 m	1 nm (lattice space)	10^{24}	2085
3-D casting	1 m	1 nm (lattice space)	10^{27}	2100

Equations of Motion: Conservation of Mass

- Turbulent flow (and fluid dynamics in general) can be mathematically described by the Navier-Stokes equations (see Bachelor, 1967 for a derivation of equations/Pope Ch 2)
- The primary goal of CFD (and LES) is to solve the discretized equations of motion.
- we use the continuum hypothesis (e.g., $\eta >>$ mean free path of molecules) so that

$$\Rightarrow \; u_i = u_i(x_j, t) \text{ and } \rho = \rho(x_j, t)$$

- **Conservation of Mass:**

$$\left.\frac{dm}{dt}\right)_{sys} = 0$$

Using Reynolds Transport Theorem (RTT, see any fluids textbook)

$$\Rightarrow \left.\frac{dm}{dt}\right)_{sys} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot d\vec{A} = 0 \;\; \Rightarrow \text{Integral form}$$

Using Gauss's theorem and shrinking the control volume to an infinitesimal size:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0 \;\; \Rightarrow \text{differential form}$$

Equations of Motion: Conservation of Momentum

Conservation of Momentum: (Newton's 2nd law)

$$\sum \vec{F} = \left.\frac{d\left(m\vec{V}\right)}{dt}\right)_{sys} \qquad \text{Using RTT} \rightarrow$$

$$\frac{\partial}{\partial t}\int_{CV} \rho \vec{V} d\forall + \int_{CS} \vec{V}\rho\vec{V}\, d\vec{A} = \underbrace{\int_{CS} \mathbf{T}\cdot\hat{n}\, d\vec{A}}_{\text{shear stress}} + \underbrace{\int_{CV} \rho\vec{b}\, d\forall}_{\text{body forces}} \Rightarrow \text{ integral form}$$

- The shear stress tensor depends on molecular processes. For a Newtonian fluid →

$$\mathbf{T} = -\left(P + \frac{2}{3}\mu\nabla\cdot\vec{V}\right)\mathbf{I} + 2\mu\mathbf{S}$$

Where $\mathbf{S} = \frac{1}{2}\left(\nabla\vec{V} + \nabla\vec{V}^T\right)$ or in index notation $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the deformation (rate of strain) tensor and **I** is the unit tensor (or identity matrix)

- The equivalent index-notation (differential) form of the momentum equation is:

$$\frac{\partial\left(\rho u_i\right)}{\partial t} + \frac{\partial\left(\rho u_i u_j\right)}{\partial x_j} = \frac{\partial}{\partial x_j}\left(2\mu S_{ij} - \frac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right) - \frac{\partial P}{\partial x_i} + \rho g_i$$

where the stress has been split into shear (viscous) and normal (pressure) components.

Equations of Motions: Conservation of Energy

Conservation of Energy: (1st law of Thermodynamics)

For a system conservation of energy is: $\dot{Q} - \dot{W} = \left.\frac{dE}{dt}\right)_{sys}$ or:
(in-out) + produced = stored

- **(in-out)** is the convective flux of energy
- **Production** is the heat conducted in + the work done on the volume (e.g., thermal flux and shear stress)

• if we use $e = c_V T$ (specific internal energy)

• and define $q_i = -k\frac{\partial T}{\partial x_i}$ as the thermal conductive flux where c_V is the specific heat and T is temperature. We can derive the following differential form for energy

$$\frac{\partial}{\partial t}(\rho E) + \frac{\partial}{\partial x_i}\left[u_i(P+E)\right] = \rho\dot{q} + \frac{\partial q_i}{\partial x_i} + \frac{\partial}{\partial x_i}\left[u_j\left(2\mu S_{ij} - \tfrac{2}{3}\mu\delta_{ij}\frac{\partial u_i}{\partial x_i}\right)\right]$$

Where the total energy is: $E = e + \tfrac{1}{2}u_i u_i$

Equations of Motion

Incompressible flow:

$$\frac{\partial u_i}{\partial x_i} = 0 \quad \boxed{\text{Conservation of Mass}}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2} + F_i \quad \boxed{\text{Conservation of Momentum}}$$

$$\frac{\partial \theta}{\partial t} + \frac{\partial u_i \theta}{\partial x_j} = \nu_\theta \frac{\partial^2 \theta}{\partial x_j^2} + Q \quad \boxed{\text{Conservation of scalar}} \text{(temp, species, etc.)}$$

$$Sc \equiv \frac{\nu}{\nu_\theta} \Rightarrow \nu_\theta = \frac{\nu}{Sc} \text{ or } \frac{\nu}{\Pr}$$

Temperature (Pr=Prandtl #)

general scalar (Sc=Schmidt #)

Dimensionless Equations of Motion

- If we nondimensionalize these equations with a velocity scale and a length scale (for example the Freestream velocity and the BL height in a boundary layer)

- We get (where the * is a nondimensional quantity):

-Conservation of Mass: $$\frac{\partial u_i^*}{\partial x_i^*} = 0$$

- Conservation of Momentum:

$$\frac{\partial u_i^*}{\partial t^*} + \frac{\partial u_i^* u_j^*}{\partial x_j^*} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re}\frac{\partial^2 u_i^*}{\partial x_j^{*2}} + F_i^*$$

where Re is based on our velocity and length scales => $Re = \frac{U_o \ell_o}{\nu}$

- For a general scalar quantity we have:

$$\frac{\partial \theta^*}{\partial t^*} + \frac{\partial u_j^* \theta^*}{\partial x_j^*} = \frac{1}{Sc\,\mathrm{Re}}\frac{\partial^2 \theta^*}{\partial x_j^{*2}} + Q^*$$

where Sc is the Schmidt number, the ratio of the diffusivity of momentum (viscosity) and the diffusivity of mass (for temperature we use the Prandtl number Pr). Sc is of order 1 (Pr for air ≈ 0.72)

Properties of the Navier-Stokes equations

• Reynolds number similarity: For a range of Re, the equations of motion can be considered invariant to transformations of scale.

• Time and space invariance: The equations are invariant to shifts in time or space. i.e., we can define the shifted space variable

$$\hat{x} = \bar{x}/L \text{ where } \bar{x} = x - X$$

$$\text{or } \hat{t} = (t - T)U/L$$

• Rotational and Reflection invariance: The equations are invariant to rotations and reflections about a fixed axis.

• Invariance to time reflections: The equations are invariant to reflections in time. They are the same going backwards or forwards in time =>

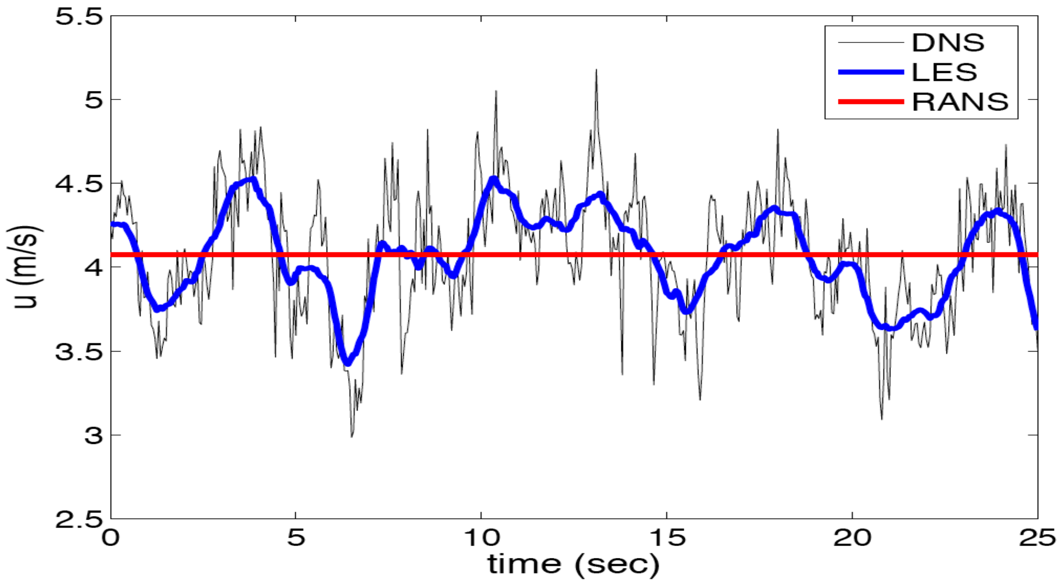
$$\hat{t} = -tU/L$$

• Galilean invariance: The equations are invariant to constant velocity translations.

$$\bar{x} = x - Vt$$

Approximating the equations of motion

- **In Numerical studies**, the equations of motion (incompressible, compressible or Boussinesq fluid) must be approximated on a computational grid
- **Three basic methodologies** are prevalent in turbulence application and research:
  - Direct Numerical Simulation (**DNS**)
    - resolve all eddies
  - Reynolds-Averaged Navier-Stokes (RANS)
    - model just ensemble statistics
  - Large-Eddy Simulation (**LES**)
    - resolve larger eddies, model smaller 'universal' ones



Some Pros and Cons of each Method

Direct Numerical Simulation (**DNS**):

- Pros
  - No turbulence model is required
  - Accuracy is only limited by computational capabilities
  - can provide reference data not available through experiments (i.e., unsteady 3D velocity and scalar fields)

- Cons
  - Restricted to low Re with relatively simple geometries
  - Very high cost in memory and computational time
  - typically "largest-possible" number of grid points is used without proper convergence evaluation.

Some Pros and Cons of each Method

Large-Eddy Simulation (**LES**):

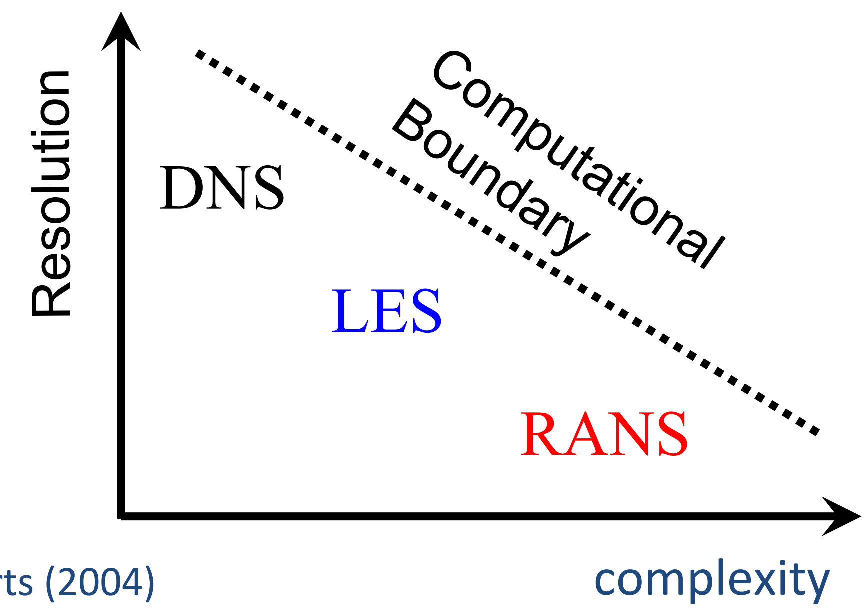
- Pros
  - Only the small scales require modeling
  - Much cheaper computational cost than DNS
  - Unsteady predictions of flow are made => gain info about extreme events in addition to the mean
  - In principle, we can gain as much accuracy as desired by refining our numerical grid
- Cons
  - Basic assumption (small scales are universal) requires independence of small (unresolved) scales from boundary conditions (especially important for flow geometry).
  - Still very costly in practical engineering applications
  - Filtering and turbulence theory of small scales still needs development for complex geometry and highly anisotropic flows

Some Pros and Cons of each Method

Reynolds Averaged Navier-Stokes (**RANS**):

- Pros
  - Low computational demand (can obtain mean stats in short time)
  - can be used in highly complex geometry
  - When combined with empirical information, can be highly useful for engineering applications

- Cons
  - Only steady flow phenomena are can be explored when taking full advantage of computational reduction
  - Models are not universal => usually pragmatic "tuning" is required for specific applications
  - More accurate turbulence models result in highly complex equation sets



Geurts (2004)

Capabilities of different simulation methods