# LES of Turbulent Flows: Lecture 2 Supplement (ME EN 7960-003)

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# Fourier Transforms

**Fourier Transforms** are a common tool in fluid dynamics (see Pope, Appendix D-G, Stull handouts online)

Some uses:

- Analysis of turbulent flow
- Numerical simulations of N-S equations
- Analysis of numerical schemes (modified wavenumbers)

- consider a periodic function $f(x)$ (could also be $f(t)$) on a domain of length 2π
- The Fourier representation of this function (or a general signal) is:

$$f(x) = \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx} \qquad (*)$$

- where $k$ is the wavenumber (frequency if $f(t)$)
- $\hat{f}_k$ are the Fourier coefficients which in general are complex

# Fourier Transforms

- why pick $e^{ikx}$ ?

  - answer: orthogonality $$\int_0^{2\pi} e^{i(k-k')x}dx = \begin{cases} 0 \text{ if } k \neq k' \\ 2\pi \text{ if } k = k' \end{cases}$$

  - a big advantage of orthogonality is independence between Fourier modes

  - $e^{ix}$ is independent of $e^{i2x}$ just like we have with Cartesian coordinates where $i,j,k$ are all independent of each other.

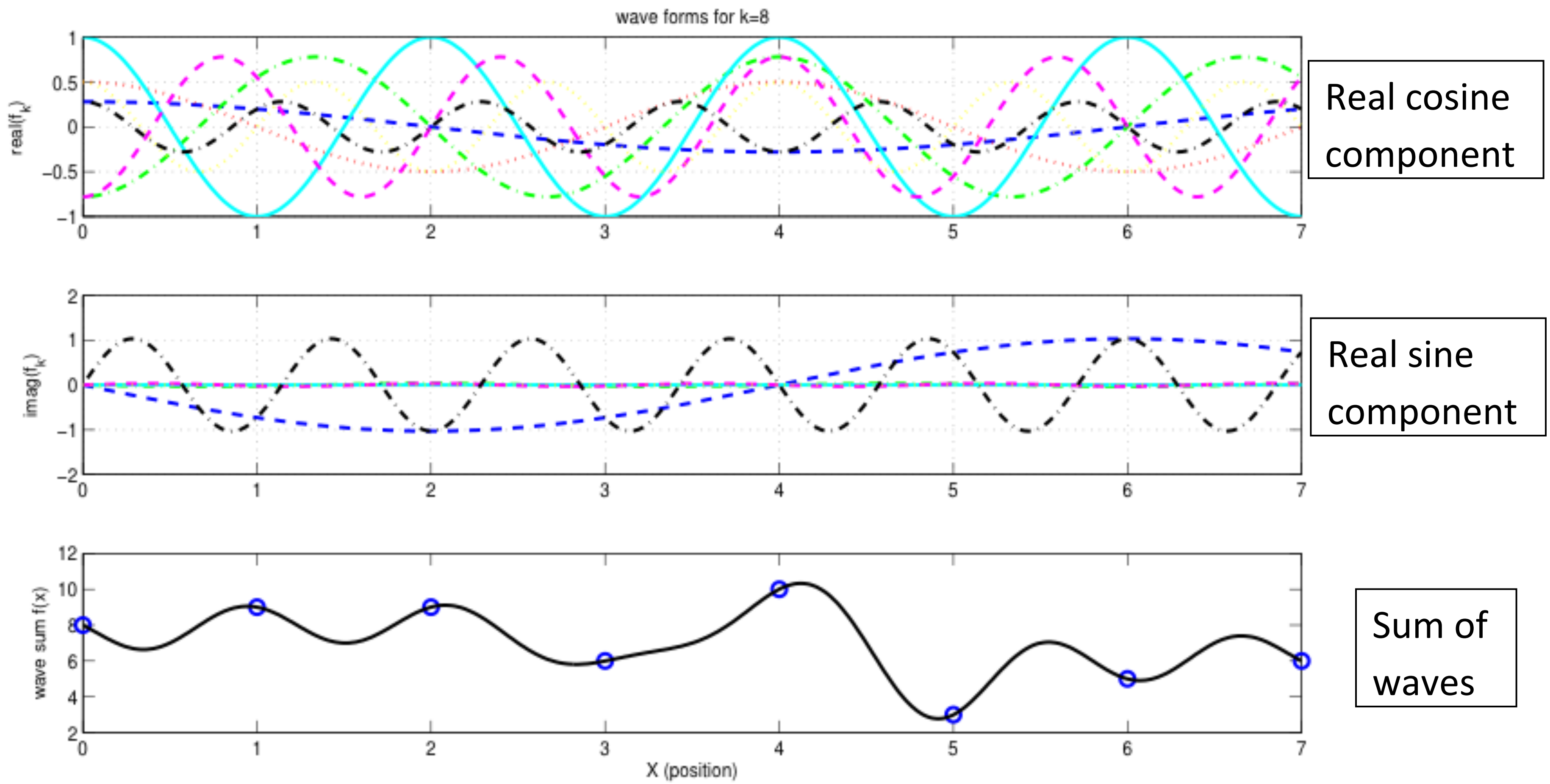
- what are we actually doing?

  - recall from Euler's formula that $e^{ix} = \cos(x) - i\sin(x)$

  - the Fourier transform decomposes a signal (space or time) into sine and cosine wave components of different amplitudes and wave numbers (or frequencies).

# Fourier Transforms

- Fourier Transform example (from Stull, 88 see example: FourierTransDemo.m)



# Fourier Transforms

- The Fourier representation given by ⊛ is a representation of a series as a function of sine and cosine waves. It takes *f(x)* and transforms it into wave space

- **<u>Fourier Transform pair</u>**: consider a periodic function on a domain of 2π

$$f_k = F\{f(x)\} \equiv \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx \quad \rightarrow \boxed{\text{forward transform}}$$

$$f(x) = F\{\hat{f}_k\}^{-1} \equiv \sum_{k=-\infty}^{k=\infty} \hat{f}_k e^{ikx} \quad \rightarrow \boxed{\text{backwards transform}}$$

- The **<u>forwards transform</u>** moves us into Fourier (or wave) space and the **<u>backwards transform</u>** moves us from wave space back to real space.

- An alternative form of the Fourier transform ⊛ (using Euler's) is:

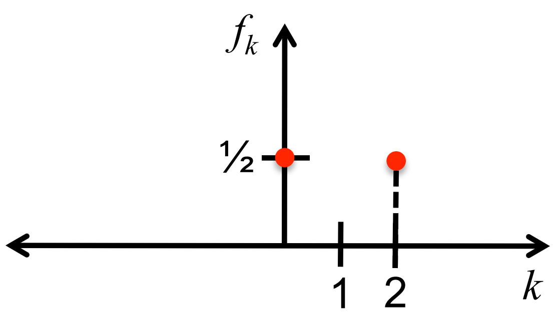
$$f(x) = a_0 + \sum_{k=1}^{k=\infty} a_k \cos(kx) - b_k \sin(kx)$$

Where $a_k$ and $b_k$ are now the real and imaginary components of $f_k$, respectively

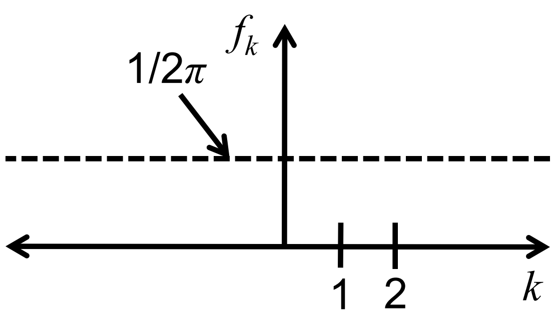
# Locality in real and wave space

- It is important to note that something that is non local in physical (real) space can be very local in Fourier space and something that is local in physical space can be non local in Fourier space.

- **Example 1**:
  $$f(x) = \cos^2 x = \frac{1+\cos 2x}{2} \quad \text{(a wave, very non local in physical space)}$$
  Fourier modes are :
  $$a_0 = 1/2, a_2 = 1/2 \text{ all other } a_k, b_k = 0$$




- **Example 2**:
  $$f(x) = \delta(x) \text{ the Dirac delta function (very non local in physical space)}$$
  $$\hat{\mathrm{f}}_{\mathrm{k}} = \frac{1}{2\pi}\int \delta(x) e^{-ikx} dx$$
  recall by definition $\int \delta(\mathrm{x-a}) f(x) dx = f(a)$
  $$\Rightarrow \int \quad = 1 \text{ for any value } k \Rightarrow \hat{f}_k = \frac{1}{2\pi}$$



# Fourier Transform Properties

1. **If f(x) is real then**:

$$\hat{f}_k = \hat{f}_k^* \quad \text{(Complex conjugate)}$$

2. **Parseval's Theorem:**

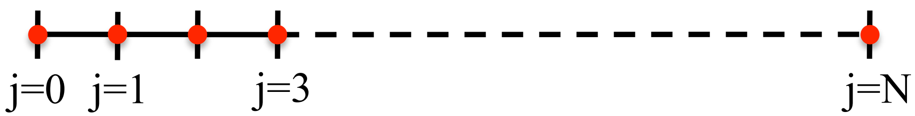
$$\frac{1}{2\pi}\int_0^{2\pi} f(x) f^*(x)\,dx = \sum_{k=-\infty}^{k=\infty} \hat{f}_k \hat{f}_k^*$$

3. **The Fourier representation is the best possible representation for f(x) in the sense that the error**:

$$\text{e} = \int_0^{2\pi} \left| f\left( x - \sum_{k=-N}^{N} c_k e^{ikx} \right) \right|^2 dx \ \text{ is minimum when } \ c_k = \hat{f}_k$$

# Discrete Fourier Transform

- Consider the periodic function $f_j$ on the domain $0 \le x \le L$




with $x_j = jh$ and $h = L/N$

Periodicity implies that $f_0 = f_N$

- <u>Discrete Fourier Representation</u>:

$$f_j = \sum_{k=-N/2}^{N/2-1} \hat{f}_k e^{i\frac{2\pi}{L}kx_j} \Rightarrow \text{ backwards (inverse) transform}$$

- we have: known: $f_j$ at $N$ points

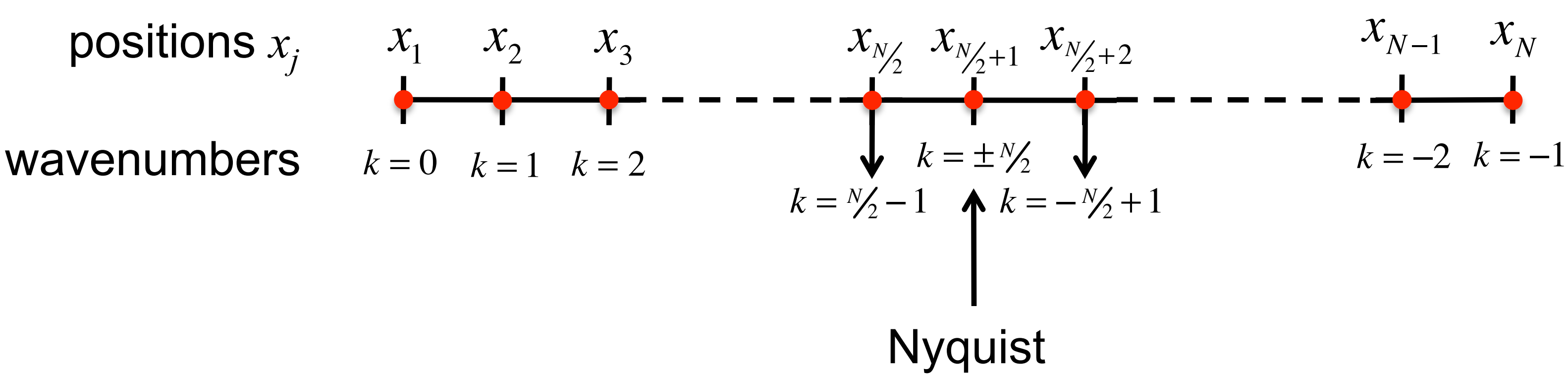
  unknown: $\hat{f}_k$ at $k$ values ($N$ of them)

- <u>Using discrete orthogonality</u>:

$$\hat{f}_k = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-i\frac{2\pi}{L}kx_j} \Rightarrow \text{ forward transform}$$

# Discrete Fourier Transform

- Discrete Fourier Transform (DFT) example and more explanation found in handouts section of website under Stull_DFT.pdf or Pope appendix F and in example: FourierTransDemo.m
- Implementation of DFT by brute force → **O**($N^2$) operations
- In practice we almost always use a Fast Fourier Transform (FFT) → **O**($N*\log_2 N$)
- Almost all FFT routines (e.g., Matlab, FFTW, Intel, Numerical Recipes, etc.) save their data with the following format:



# Fourier Transform Applications

Autocorrelation:

- We can use the discrete Fourier Transform to speed up the autocorrelation calculation (or in general any cross-correlation with a lag).

-Discretely $R_{ff}(s_l) = \sum_{j=0}^{N-1} f(x_j) f(x_j + s_l)$ this is $\mathbf{O}(\mathrm{N}^2)$ operations

correlation with itself

-if we express $R_{ff}$ as a Fourier series

$$R_{ff}(s_l) = \sum_k \hat{R}_{ff} e^{iks_l} \Rightarrow R_{ff}(0) = \sum_k \hat{R}_{ff} \text{ and we can show that } R_{ff}(0) = \sum_k N\left|\hat{f}_k\right|^2$$

magnitude of the Fourier coefficients

-how can we interpret this??

-In physical space

$$R_{ff}(0) = \sum_{j=0}^{N-1} f_j^2 \text{ (i.e. the mean variance)} \Rightarrow \sum_{j=0}^{N-1} f_j^2 = \sum_{k=-N/2}^{N/2-1} N\left|\hat{f}_k\right|^2$$

total contribution to the variance

energy spectral density

# Fourier Transform Applications

Energy Spectrum: (power spectrum, energy spectral density)

- If we look at specific $k$ values from our autocorrelation function we have:

$$E(k) = N\left|\hat{f}_k\right|^2$$

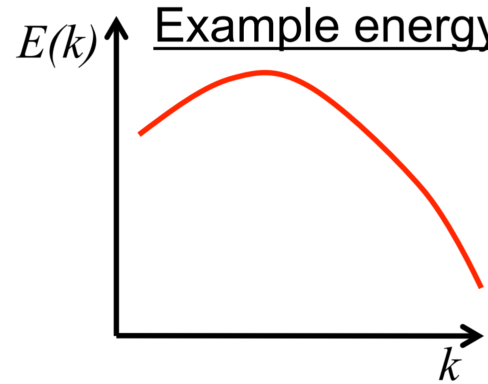
where $E(k)$ is the energy spectral density

The square of the Fourier coefficients is the contribution to the variance by fluctuations of scale $k$ (wavenumber or equivalently frequency

- Typically (when written as) $E(k)$ we mean the contribution to the turbulent kinetic energy (tke) = $\frac{1}{2}(u^2+v^2+w^2)$ and we would say that $E(k)$ is the contribution to tke for motions of the scale (or size) $k$ . For a single velocity component in one direction we would write $E_{11}(k_1)$.

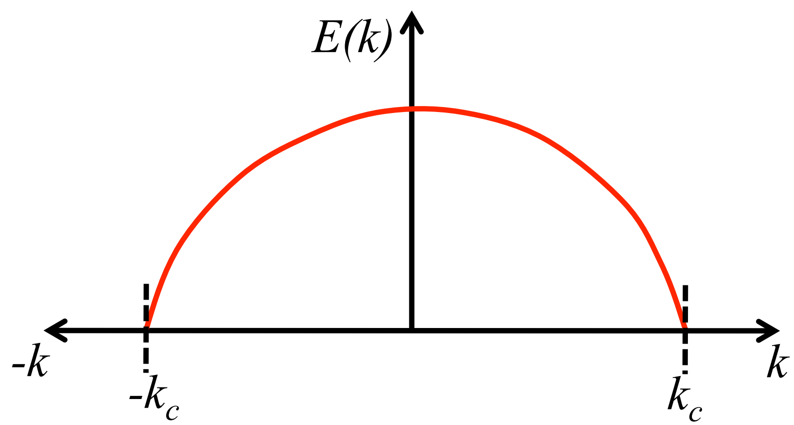
- This means that the energy spectrum and the autocorrelation function form a Fourier Transform pair (see Pope for details)

Example energy spectrum



# Sampling Theorem

Band-Limited function: a function where $\hat{f}_k = 0 \text{ for } |k| > k_c$




Theorem: If $f(x)$ is band limited, i.e., $\hat{f}_k = 0 \text{ for } |k| > k_c$, then $f(x)$ is completely represented by its values on a discrete grid, $x_n = n\pi / k_c$ where $n$ is an integer ($-\infty < n < \infty$) and $k_c$ is called the Nyquist frequency.

Implication:

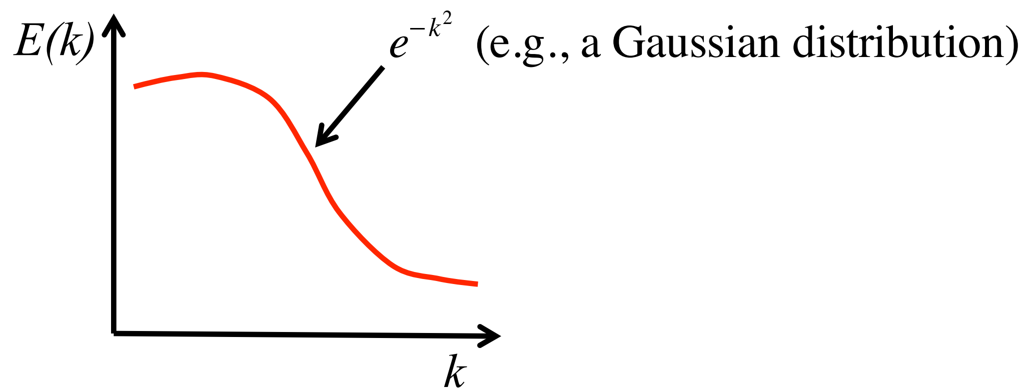
- If we have $x_j = j\, \pi / k_c = jh \Rightarrow h = \pi / k_c$ with a domain of 2π: $h = 2\pi / N = \pi / k_c \Rightarrow k_c = N/2$

➔ If the number of points is ≥ $2k_c$ then the discrete **Fourier Transform=exact solution**

e.g., for $f(x)=\cos(6x)$ we need $N \geq 12$ points to represent the function exactly

# Sampling Theorem

- What if f(x) is not band-limited?




- or $f(x)$ is band limited but sampled at a rate < $k_c$, for example $f(x)=cos(6x)$ with 8 points.

- **Result**: **Aliasing** → contamination of resolved energy by energy outside of the resolved scales.

# Aliasing

- Consider: $e^{ik_1 x_j}$ and $e^{ik2x_j}$ and let $k_1 = k_2 + 2mk_c$

  where $k_c =$ Nyquist frequency, $m = \pm$ any integer value and $x_j = j\,\pi/k_c$

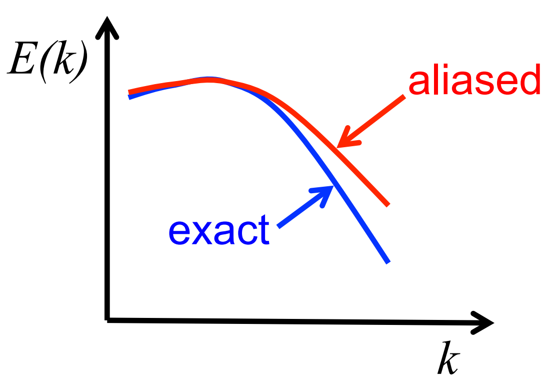
  $$e^{ik_1 x_j} = e^{i(k_2 + 2mk_c)x_j}$$

  $$= e^{ik_2 x_j} e^{2mk_c x_j} = e^{ik_2 x_j} e^{2mk_c j \pi/k_c}$$

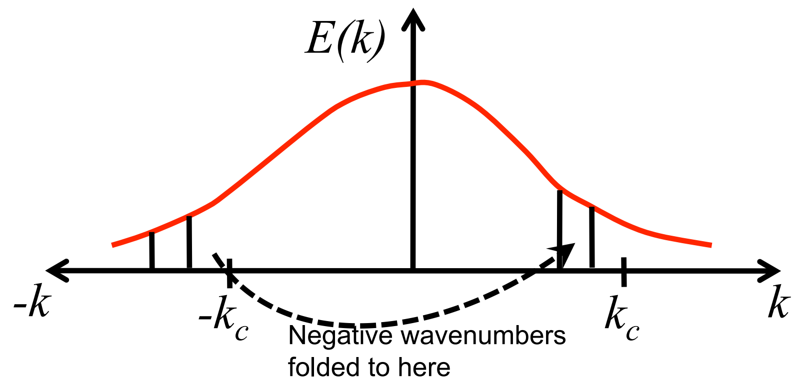
  $$= e^{ik_2 x_j} \underbrace{e^{i2\pi mj}}_{=1 \text{ (integer function of } 2\pi)}$$

  $e^{ik_1 x_j} = e^{ik_2 x_j} \Rightarrow$ result is that we can't tell the difference between $k_2$ and $k_1 = k_2 + 2mk_c$ on a discrete grid. $k_1$ is aliased onto $k_2$

**What does this mean for spectra?**



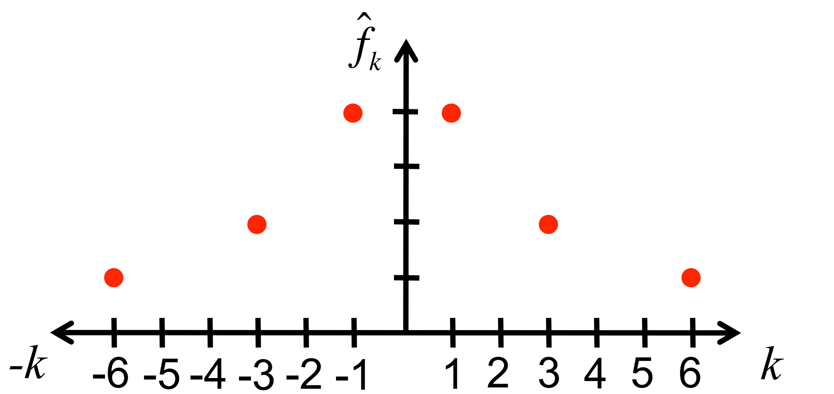

**What is actually happening to *E(k)*?**



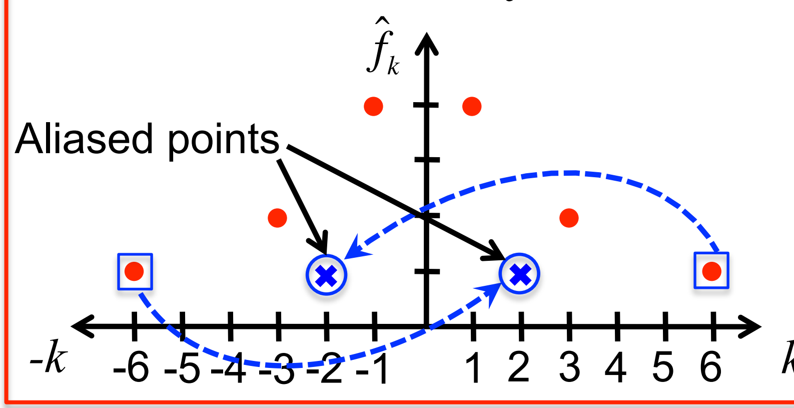
# Aliasing Example

Consider a function: $\text{e.g.,} f(x) = \cos(x) + \frac{1}{2}\cos(3x) + \frac{1}{4}\cos(6x)$

Fourier coefficients (all real since only cosine)




Consider N=8 ➔ $k_c = N/2 = 4$




$$\text{Aliasing } \Rightarrow k_1 = k_2 + 2mk_c = k_2 + 8m \Rightarrow -6 \text{ gets aliased to } 2$$

$$\text{and if } m = -1 \Rightarrow k_1 = k_2 - 8 \Rightarrow 6 \text{ gets aliased to -2}$$

Aliasing ⬇ if N ⬆

For more on Fourier Transforms see Pope Ch. 6, online handout from Stull or Press et al., Ch 12-13.