LES of Turbulent Flows: Lecture 1 Supplement (ME EN 7960-003)

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Statistical Tools for Turbulent Flow

- A consequence of the random behavior of turbulence and the fact that it is the histogram that appears to be reproducible is that turbulence is that it is usually studied from a statistical viewpoint.

Probability:

Some event (value) V_b in the space V (e.g., our sample velocity field)

$$P = P(B) = P\{U < V_b\} \quad \text{for event } B \equiv \{U < V_b\}$$

this is the probablity (likely-hood) that U is les than V_b where P=0 means there is no chance and P=1 means we have certainty.

Cumulative density function (cdf):

$$F(V) \equiv P\{U < V\} \text{ where for a specific event } P(B) = P\{U < V_b\} = F(V_b)$$

$$\text{and if we have } P(C) \equiv P\{V_a \le U < V_b\} = F(V_b) - F(V_a)$$

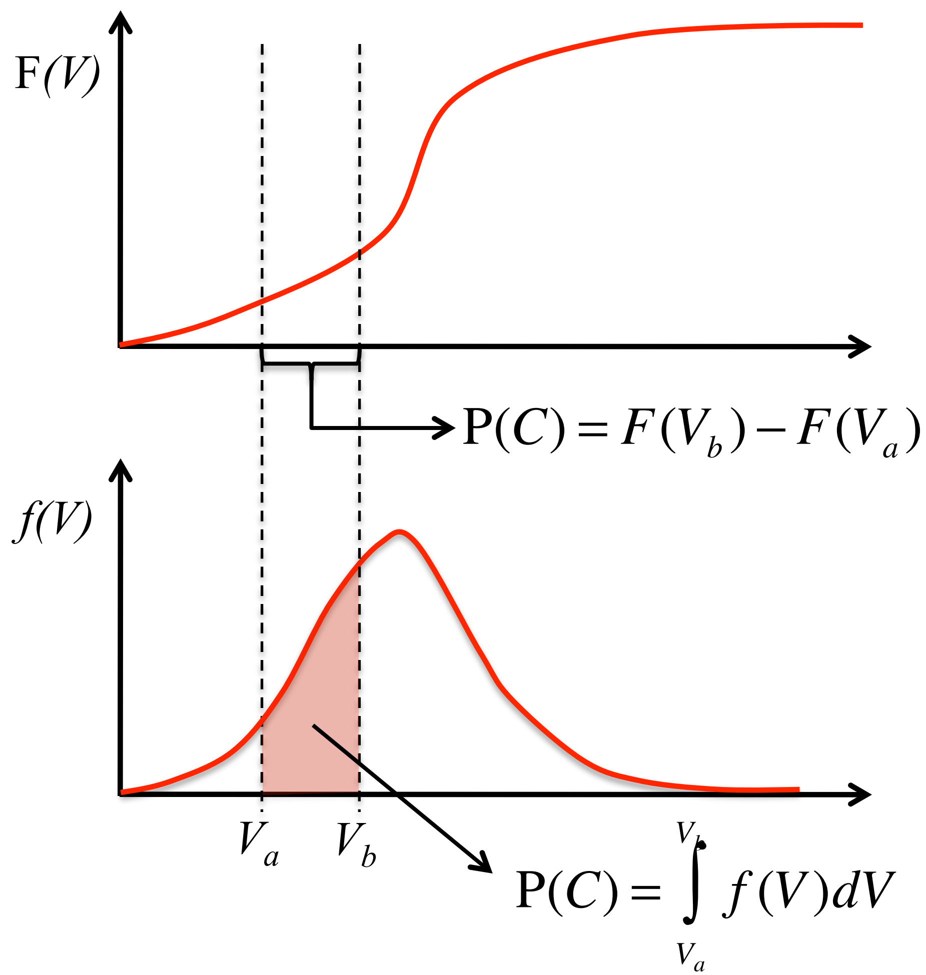
$$\text{this is bounded by: } \mathrm{F}(-\infty) = 0 \ (U < -\infty \text{ is impossible})$$

$$\mathrm{F}(\infty) = 1 \ (U < \infty \text{ is certain})$$

We also know that since P is non negative, F is non a non decreasing function

Statistical Tools for Turbulent Flow

pdf vs cdf



Means and Moments

- The pdf fully defines the statistics of a signal (random variable)
- If two signals have the same pdf, they are considered to be statistically identical
- We can also define a signal by its individual stats that collectively describe the pdf

  - <u>The mean (or expected value)</u>

$$\langle U \rangle \equiv \int_{-\infty}^{\infty} Vf(V)dV \text{ in a discrete form we have } \frac{1}{N}\sum_{i=1}^{N} V_i$$

  - The mean is the probability weighted sum of all possible values
  - In general for and $Q(U)$ ➔ something that is a function of U

$$\langle Q(U) \rangle \equiv \int_{-\infty}^{\infty} Q(V)f(V)dV$$

  and from this equation we can show that for constants a and b:

$$\langle aQ(U) + bR(U) \rangle = a\langle Q(U) \rangle + b\langle R(U) \rangle$$

Means and Moments

- We can also define a fluctuation from the mean by

$$u \equiv U - \langle U \rangle$$

- The variance is then the mean square fluctuation

$$\sigma_u^2 = \mathrm{var}(U) = \langle u^2 \rangle \equiv \int_{-\infty}^{\infty} \left(V - \langle U \rangle\right)^2 f(V)dV \;\; \text{discretely} \;\; \sigma_u^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(V_i - \langle U \rangle\right)^2$$

- And the standard deviation (or rms) is simply the root of the variance

$$\sigma_u = sdev(U) = \langle u^2 \rangle^{1/2}$$

- We can define the n^{th} central moment as:

$$\mu_n \equiv \langle u^n \rangle = \int_{-\infty}^{\infty} \left(V - \langle U \rangle\right)^n f(V)dV$$

- Many times we prefer to express variables as standardized random variables

$$\hat{U} \equiv \frac{U - \langle U \rangle}{\sigma_u} \;\; \text{a centered and scaled variable}$$

- The standardized moments are then

$$\hat{\mu}_n \equiv \frac{\mu_n}{\sigma_u^n} = \int_{-\infty}^{\infty} \hat{V}^n \hat{f}(\hat{V})d\hat{V} \;\; \text{with} \;\; \hat{\mu}_0 = 1, \hat{\mu}_1 = 0 \;\text{and}\; \hat{\mu}_2 = 1$$

Means and Moments

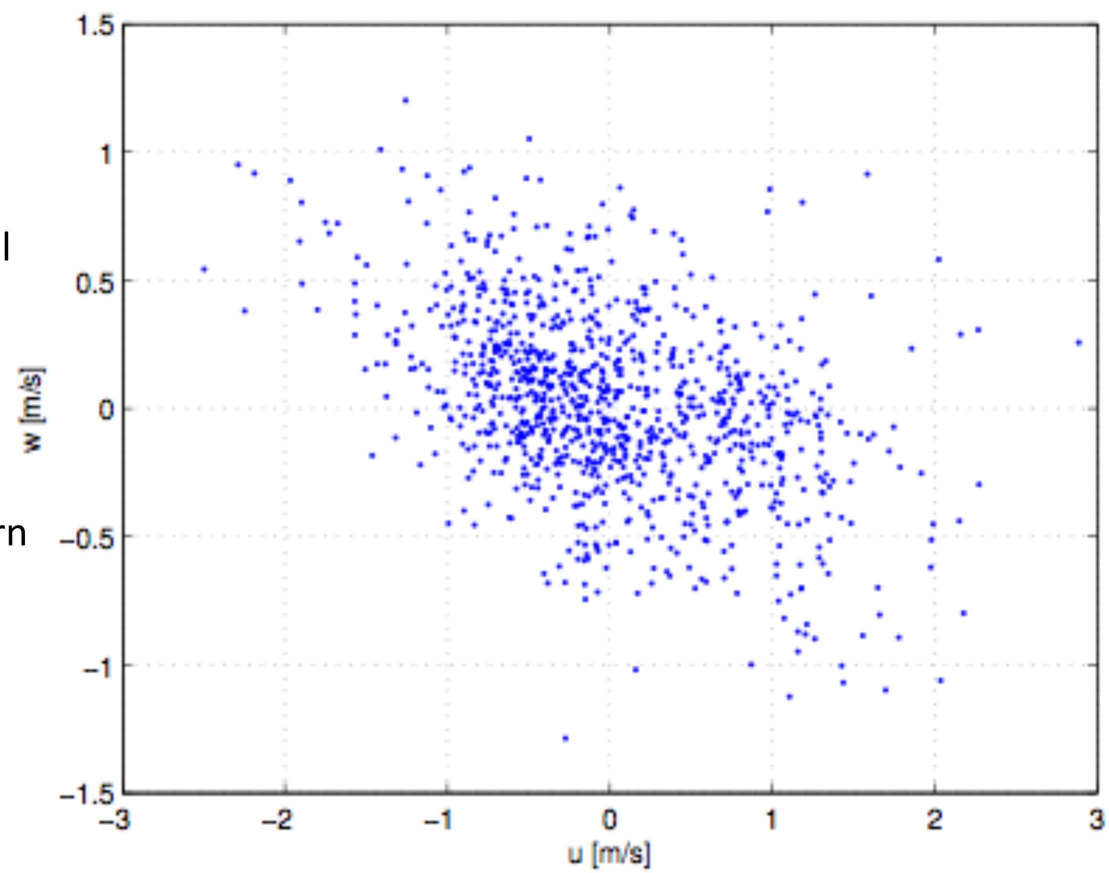
- The different moments each describe an aspect of the shape of the pdf

$$\mu_1 \Rightarrow \text{ mean or expected value}$$
$$\mu_2 \Rightarrow \text{ variance}$$
$$\mu_3 \Rightarrow \text{ skewness}$$
$$\mu_4 \Rightarrow \text{ kurtosis (or flatness)}$$

- In basic probability theory we have several different types of pdfs. Pope 3.3 gives a fairly extensive list of these the most important of which is the normal or Gaussian distribution

Joint Random Variables

- So far the description has been limited to single Random variables but turbulence is governed by the Navier-Stokes equations which are a set of 3 coupled PDEs.
- We expect this will result in some correlation between different velocity components
- Example, turbulence data from the ABL: scatter plot of horizontal (u) and vertical (w) velocity fluctuations.
- The plot appears to have a pattern (negative slope)



Joint Random Variables

- <u>Joint Cumulative Density Function (joint CDF)</u>:

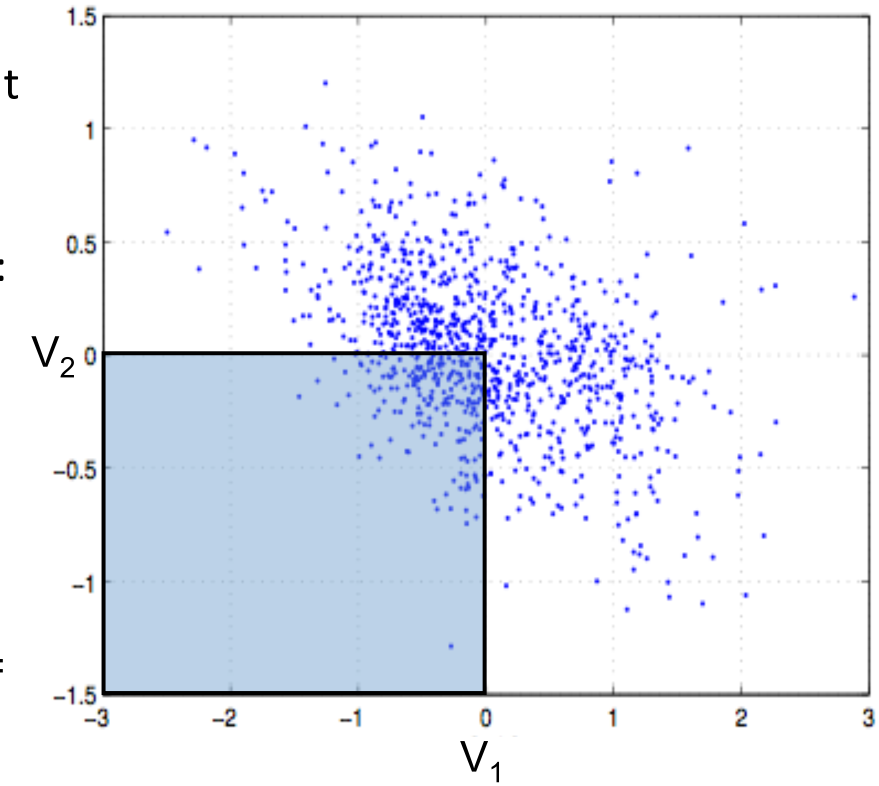
$$F_{12}(V_1, V_2) \equiv P\{U_1 < V_1, U_2 < V_2\}$$

Sample space of our random variables U_1 and U_2

- In the figure, the CDF is the probability that the variable (U_1 and U_2) lie within the shaded region
- The joint CDF has the following properties:
  - It is non decreasing
  - $F_{12}(-\infty, V_2) = P\{U_1 < -\infty, U_2 < V_2\} = 0$
  - $F_{12}(\infty, V_2) = P\{U_1 < \infty, U_2 < V_2\} =$

$P\{U_2 < V_2\} = F_2(V_2)$

i.e. since $U_1 < \infty$ is certain, the joing cdf $=$ the single variable cdf



Joint PDF

- The **joint PDF** is given by:

$$f_{12}(V_1,V_2) = \frac{\partial^2}{\partial V_1 \partial V_2} F_{12}(V_1,V_2)$$

- Similar to the single variable PDF, if we integrate over V_1 and V_2 we get the probability

$$P\{V_{1a} \le U_1 < V_{1b}, V_{2a} \le U_2 < V_{2b}\} = \int_{V_{1a}}^{V_{1b}} \int_{V_{2a}}^{V_{2b}} f_{12}(V_1,V_2)\,dV_2\,dV_1$$

- The joint PDF has the following properties:
  - $f_{12}(V_1,V_2) \ge 0$
  - $f_2(V_2) = \int_{-\infty}^{\infty} f_{12}(V_1,V_2) \Rightarrow$ the marginal PDF of U_2
  - $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{12}(V_1,V_2)\,dV_1\,dV_2 = 1$
- Similar to a single variable, if we have Q(U_1, U_2) then

$$\langle Q(U_1,U_2) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(V_1,V_2) f_{12}(V_1,V_2)\,dV_2\,dV_1$$

- With this idea we can give a rigorous definition for a few important stats (next)

Important single point stats for joint variables

- **covariance**:

$$\operatorname{cov}(U_1,U_2)=\langle u_1 u_2\rangle=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(V_1-\langle U_1\rangle)(V_2-\langle U_2\rangle)f_{12}(V_1,V_2)\,dV_2\,dV_1$$

- Or for discrete data

$$\operatorname{cov}(U_1,U_2)=\langle u_1 u_2\rangle=\frac{1}{N-1}\sum_{j=1}^{N}(V_{1j}-\langle U_1\rangle)(V_{2j}-\langle U_2\rangle)$$

- We can also define the correlation coefficient (non dimensional)

$$\rho_{12}=\frac{\langle u_1 u_2\rangle}{\left[\langle u_1^2\rangle\langle u_2^2\rangle\right]^{1/2}}$$

  - Note that -1 ≤ ρ_{12} ≤ 1 and negative value mean the variables are anti-correlated with positive values indicating a correlation

- **Practically speaking**, we find the PDF of a time (or space) series by:
  1. Create a histogram of the series (group values into bins)
  2. Normalize the bin weights by the total # of points

Two-point statistical measures

- **autocovariance:** measures how a variable changes (or the correlation) with different lags

$$R(s) \equiv \langle u(t)u(t+s) \rangle$$

- or the autocorrelation function

$$\rho(s) \equiv \langle u(t)u(t+s) \rangle / \langle u(t)^2 \rangle$$

  - These are very similar to the covariance and correlation coefficient
  - The difference is that we are now looking at the linear correlation of a signal with itself but at two different times (or spatial points), i.e. we lag the series.

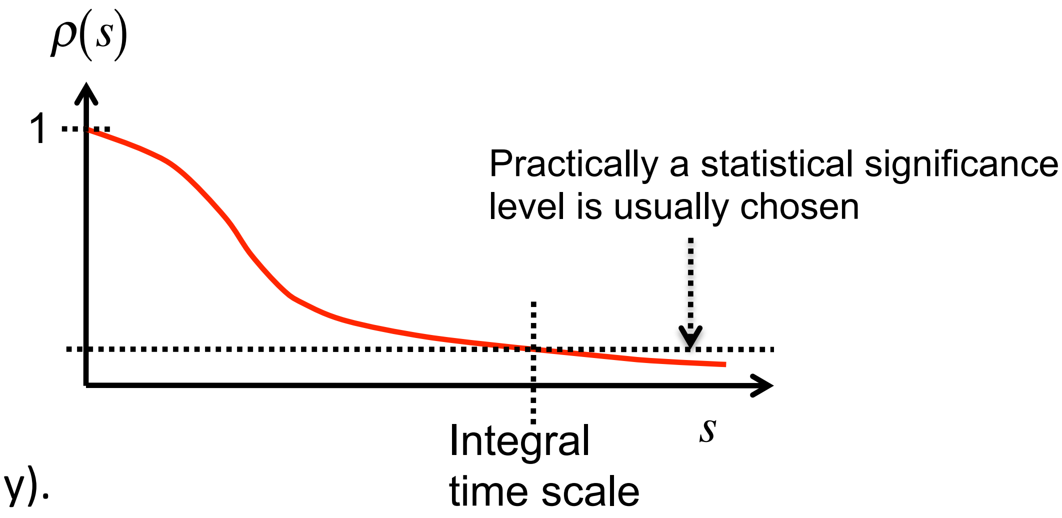
- Discrete form of autocorrelation:

$$\rho\left(s_j\right) = \frac{\sum_{k=0}^{N-j-1}\left(u_k u_{k+j}\right)}{\sum_{k=0}^{N-1}\left(u_k^2\right)}$$

- We could also look at the cross correlations in the same manner (between two different variables with a lag).
- Note that: $\rho(0) = 1$ and $|\rho(\mathrm{s})| \leq 1$

Two-point statistical measures

- In turbulent flows, we expect the correlation to diminish with increasing time (or distance) between points:



- We can use this to define an Integral time scale (or space). It is defined as the time lag where the integral $\int_0^\infty \rho(s)ds$ converges. and can be used to define the largest scales of motion (statistically).

- Another important 2 point statistic is the **structure function**:

$$D_n(r) \equiv \left\langle \left[U_1(x+r,t) - U_1(x,t)\right]^n \right\rangle$$

This gives us the average difference between two points separated by a distance r raised to a power n. In some sense it is a measure of the moments of the velocity increment PDF. Note the difference between this and the **autocorrelation which is statistical linear correlation** (ie multiplication) of the two points.