2.8 Summation Notation

In the last section we encountered heat fluxes with three components and momentum fluxes with nine components. It is very laborious to write a separate forecast equation for each of these nine fluxes, but seemingly necessary if we want to better understand the boundary layer.

To ease the burden, we can employ a shorthand notation known as Einstein's summation notation. With just one term, we can represent all nine of the momentum fluxes. In this section, we first define some terms, then state rules with some examples, and finally show how summation notation and vector notation are related.

2.8.1 Definitions and Rules

Let m, n, and q be integer variable indices that can each take on the values of 1, 2, or 3. Let A_m represent a generic velocity vector, X_m represent a generic component of distance, and δ_m represent a generic unit vector (a vector of length unity and direction in one of the three Cartesian directions). By using indices as subscripts to these generic variables, we can define:

m = 1, 2, or 3	$A_1 = U$	$X_1 = x$
n = 1, 2, or 3	$A_2 = V$	$X_2 = y$
q = 1, 2, or 3	$A_3 = W$	$X_3 = z$

A variable with:	no free (unsummed) indices	= scalar
	1 free index	= vector
	2 free indices	= tensor

Unit vectors: $\delta_1 = \mathbf{i}$ $\delta_2 = \mathbf{j}$ $\delta_3 = \mathbf{k}$

Physically, we expect that some forces act in all directions while others might act in just one or two. To be able to isolate such directional dependence, we must define two new terms with unusual characteristics:

Kronecker Delta (a scalar quantity even though it has two indices):

$$\delta_{mn} = \begin{cases} +1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \tag{2.8.1a}$$

Alternating Unit Tensor (a scalar even though it has three indices):

$$\varepsilon_{mnq} = \begin{cases} +1 & \text{for } mnq = 123, 231, \text{ or } 312 \\ -1 & \text{for } mnq = 321, 213, \text{ or } 132 \\ 0 & \text{for any two or more indices alike} \end{cases} \tag{2.8.1b}$$

The unit vector, δ_m, and the Kronecker delta, δ_{mn}, can easily be confused. They represent distinctly different quantities that are not interchangeable. To help distinguish between these two quantities, remember that the Kronecker delta is a scalar and always has two subscripts, while the unit vector is a vector and always has just one subscript.

Three fundamental rules apply within summation notation: two concern repeated indices within any one term, and the other concerns nonrepeated (free) indices.

Rule (a): Whenever two identical indices appear in the same one term, it is implied that there is a sum of that term over each value (1, 2, and 3) of the repeated index.

Rule (b): Whenever one index appears unsummed (free) in a term, then that same index must appear unsummed in all terms in that equation. Hence, that equation effectively represents 3 equations for each value of the unsummed index. This insures that all terms are tensorally consistent with the other terms in the equation.

Rule (c): The same index cannot appear more than twice in one term.

2.8.2 Examples

Problem 1 and Solution, demonstrating Rule (a).

$$A_n \frac{\partial B_m}{\partial X_n} = A_1 \frac{\partial B_m}{\partial X_1} + A_2 \frac{\partial B_m}{\partial X_2} + A_3 \frac{\partial B_m}{\partial X_3}$$

$$= U \frac{\partial B_m}{\partial x} + V \frac{\partial B_m}{\partial y} + W \frac{\partial B_m}{\partial z}$$

Problem 2 and Solution, demonstrating Rule (a).

$$\delta_{2n} A_n = \delta_{21} A_1 + \delta_{22} A_2 + \delta_{23} A_3$$

$$= 0 + A_2 + 0$$

$$= V$$

The latter example leads to an important general conclusion:

$$\delta_{mn} A_n = A_m \qquad (2.8.2a)$$

namely, the Kronecker delta changes the index of A from n to m.

Problem 3, demonstrating Rule (b): Given the following equation, expand it:

$$A_m = B_m + \delta_{mn} C_n$$

Solution to 3. In each term is the same unrepeated index, m. Thus, this equation represents three equations:

$$\begin{cases} A_1 = B_1 + \delta_{1n} C_n \\ A_2 = B_2 + \delta_{2n} C_n \\ A_3 = B_3 + \delta_{3n} C_n \end{cases}$$

The last term in each equation has the Kronecker delta, which means we can use the general conclusion above to yield:

$$\begin{cases} A_1 = B_1 + C_1 \\ A_2 = B_2 + C_2 \\ A_3 = B_3 + C_3 \end{cases}$$

Problem 4, demonstrating all rules: One form of the equation of motion is written here in summation notation. For now, just accept this equation as a given example; we will discuss the physics of it in more detail in the next chapter. This equation employs repeated and nonrepeated indices, the Kronecker delta, the alternating unit tensor, and the stress tensor τ (to be discussed in the next section). Let both A and B represent velocities. This is quite a complex example, which you should study carefully:

$$\frac{\partial A_m}{\partial t} + B_n \frac{\partial A_m}{\partial X_n} = -\delta_{m3}\, g + f_c\, \varepsilon_{mn3}\, B_n - \frac{1}{\rho}\frac{\partial p}{\partial X_m} + \frac{1}{\rho}\left[\frac{\partial \tau_{mn}}{\partial X_n}\right] \quad (2.8.2b)$$

Using the previous rules and definitions, we can step-by-step expand the shorthand equation above to discover the equivalent set of equations written in more conventional form.

Solution to 4. First, sum over repeated indices:

$$\frac{\partial A_m}{\partial t} + B_1 \frac{\partial A_m}{\partial X_1} + B_2 \frac{\partial A_m}{\partial X_2} + B_3 \frac{\partial A_m}{\partial X_3} = -\delta_{m3}\, g + f_c\, \varepsilon_{m13}\, B_1 + f_c\, \varepsilon_{m23}\, B_2$$

$$+ f_c\, \varepsilon_{m33}\, B_3 - \frac{1}{\rho}\frac{\partial p}{\partial X_m} + \frac{1}{\rho}\left[\frac{\partial \tau_{m1}}{\partial X_1} + \frac{\partial \tau_{m2}}{\partial X_2} + \frac{\partial \tau_{m3}}{\partial X_3}\right]$$

The term with ε_{m33} becomes zero because of the repeated index in the alternating unit tensor.

Next, write a separate equation for each value of the free index, m:

For m = 1:

$$\frac{\partial A_1}{\partial t} + B_1 \frac{\partial A_1}{\partial X_1} + B_2 \frac{\partial A_1}{\partial X_2} + B_3 \frac{\partial A_1}{\partial X_3} = -\delta_{13}\, g + f_c\, \varepsilon_{113}\, B_1 + f_c\, \varepsilon_{123}\, B_2$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial X_1} + \frac{1}{\rho} \left[\frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + \frac{\partial \tau_{13}}{\partial X_3} \right]$$

In this equation the terms with δ_{13} and ε_{113} are both zero. We will leave out similar terms in the equations for the remaining components. The factor $\varepsilon_{123} = 1$.

For m = 2:

$$\frac{\partial A_2}{\partial t} + B_1 \frac{\partial A_2}{\partial X_1} + B_2 \frac{\partial A_2}{\partial X_2} + B_3 \frac{\partial A_2}{\partial X_3} = + f_c\, \varepsilon_{213}\, B_1$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial X_2} + \frac{1}{\rho} \left[\frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + \frac{\partial \tau_{23}}{\partial X_3} \right]$$

The factor ε_{213} in the equation above equals -1.

For m = 3:

$$\frac{\partial A_3}{\partial t} + B_1 \frac{\partial A_3}{\partial X_1} + B_2 \frac{\partial A_3}{\partial X_2} + B_3 \frac{\partial A_3}{\partial X_3} = - \delta_{33}\, g$$

$$- \frac{1}{\rho} \frac{\partial p}{\partial X_3} + \frac{1}{\rho} \left[\frac{\partial \tau_{31}}{\partial X_1} + \frac{\partial \tau_{32}}{\partial X_2} + \frac{\partial \tau_{33}}{\partial X_3} \right]$$

The factor $\delta_{33} = 1$ in the equation above.

After substituting U for A_1, y for X_2, τ_{zx} for τ_{31}, etc., we finally get:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = +f_c V - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right]$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -f_c U - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\left[\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right]$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left[\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right]$$

(2.8.2c)

Discussion of Problem 4. Comparing the above set of equations to the original shorthand version, we begin to appreciate the power of Einstein's summation notation. This tool will be used throughout the remaining chapters. It is analogous to vector notation, which is examined in the next section.

Usually, the shorthand (summation) form of (2.8.2b) is written as follows:

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = -\delta_{i3}\, g + f_c\, \varepsilon_{ij3}\, U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\left[\frac{\partial \tau_{ij}}{\partial x_j}\right] \qquad (2.8.2d)$$

where vectors like U_i have three components (U, V, W), and where (i, j, k) are indices, not unit vectors. This is the form that will be used for the remainder of the text.

Finally, we should recognize that a whole term is a scalar, vector, or tensor if the term has zero, one, or two unsummed (free) variable indices, respectively. For example, the $f_c\, \varepsilon_{ij3}\, U_j$ term in the equation above is a vector, because there is only one unsummed variable index, i.

2.8.3 Comparison with Vector Notation

Vectors can represent three Cartesian components, and tensors can represent nine. There is a one-to-one correspondence between vector definitions and Einstein's summation notation, as might be expected. The following is an explanation of how vectors can be rewritten in summation notation, and how vector operations such as the dot and cross product can be represented. In these examples, vector operations apply only to the vector parts of each term; scalar parts can be separated and interpreted as simple products.

The definitions below relate basic vector forms (especially, unit vectors) and operators to summation notation:

Vector: $\mathbf{A} \equiv A_m \boldsymbol{\delta}_m$ (2.8.3a)

Dot Product: $\boldsymbol{\delta}_m \bullet \boldsymbol{\delta}_n \equiv \delta_{mn}$ (2.8.3b)

Cross Product: $\boldsymbol{\delta}_m \times \boldsymbol{\delta}_n \equiv \varepsilon_{mnq} \boldsymbol{\delta}_q$ (2.8.3c)

Del Operator: $\nabla(\;) \equiv \boldsymbol{\delta}_m \dfrac{\partial(\;)}{\partial X_m}$ (2.8.3d)

Examples are presented here of other vector operations using the definitions above. First, consider the dot product between two vectors:

$$\begin{aligned}\mathbf{A} \bullet \mathbf{B} &= (\boldsymbol{\delta}_m A_m) \bullet (\boldsymbol{\delta}_n B_n) \\ &= (\boldsymbol{\delta}_m \bullet \boldsymbol{\delta}_n)\, A_m B_n \\ &= \delta_{mn} A_m B_n \\ &= A_m B_m\end{aligned} \qquad (2.8.3e)$$

On the first line, we substituted each vector by its summation notation as three Cartesian components times their respective component magnitudes. On the second line, the vector (boldface) terms were grouped together, leaving the product of the magnitudes remaining at the end. Then, the definition of a vector dot product was used to substitute the Kronecker delta. Finally, the Kronecker delta was used to change one subscript to equal the other. The end result, $A_m B_m = A_1 B_1 + A_2 B_2 + A_3 B_3$, is indeed a scalar that is equal in value to the scalar result of the vector dot product.

A similar development can be made for the cross product of two vectors:

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (\boldsymbol{\delta}_m A_m) \times (\boldsymbol{\delta}_n B_n) \\ &= (\boldsymbol{\delta}_m \times \boldsymbol{\delta}_n)\, A_m B_n \\ &= \varepsilon_{mnq} A_m B_n \boldsymbol{\delta}_q\end{aligned} \qquad (2.8.3f)$$

The result is a vector, as is required for a cross product between two vectors. The reader can perform the implied sums to verify that the expected terms are obtained. Note that $\varepsilon_{mnq} = \varepsilon_{qmn} = \varepsilon_{nqm}$, resulting in equivalent but different-looking expressions for (2.8.3f).

As a final example, we will look at the divergence of a vector:

$$\nabla \bullet A = \left(\delta_m \frac{\partial}{\partial X_m}\right) \bullet (\delta_n A_n)$$
$$= (\delta_m \bullet \delta_n) \frac{\partial A_n}{\partial X_m}$$
$$= (\delta_{mn}) \frac{\partial A_n}{\partial X_m}$$
$$= \frac{\partial A_m}{\partial X_m} \qquad (2.8.3g)$$

We will have little use for vector notation in the remainder of this book, because summation notation is frequently easier to use. This section was presented only because most meteorologists are familiar with vector notation from their studies of atmospheric dynamics.

2.9 Stress

We have seen that the covariance statistic describes a turbulent flux. But a momentum flux is analogous to a stress. In this section, we review the nature of stress and relate it to various turbulence statistics.

Stress is the force tending to produce deformation in a body. It is measured as a force per unit area. Three types of stress appear frequently in studies of the atmosphere: pressure, Reynolds stress, and viscous shear stress.

2.9.1 Pressure

Pressure is a type of stress that can act on a fluid at ***rest***. For an infinitesimally small fluid element, such as idealized as the cube sketched in Fig 2.16a, pressure acts equally in all directions. ***Isotropic*** is the name given to characteristics that are the same in all directions (see Figs 2.4 and 2.5).

If we consider just one face of this cube, as in Fig. 2.16b, we see that the isotropic nature of pressure tends to counteract itself in all directions except in a direction normal to (perpendicular to) the surface of the cube. Forces acting normal to all faces of the cube tend to compress or expand the cube, thereby deforming it (Fig 2.16c).

At sea level, the standard atmospheric pressure is 1.013×10^5 N/m^2. A pascal (Pa) is defined as 1 N/m^2, thus standard atmospheric pressure at sea level is 101.3 kPa. Historically, millibars (mb) have also been used as a pressure unit, where 1 mb = 100 pascals. In kinematic units, standard sea-level pressure is 82714 m^2/s^2. Although this value is much larger than the other stresses to be discussed next, it is almost totally counteracted by the influence of gravity, as described by the hydrostatic approximation.