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ANALYTICAL METHODS FOR THE DEVELOPMENT OF REYNOLDS-STRESS CLOSURES IN TURBULENCE

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INTRODUCTION

Despite over a century of research, turbulence remains the major unsolved problem of classical physics. While most researchers agree that the basic physics of turbulence can be described by the Navier-Stokes equations, limitations in computer capacity make it impossible—for now and the foreseeable future—to directly solve these equations for the complex turbulent flows of technological interest. Hence, virtually all scientific and engineering calculations of nontrivial turbulent flows, at high Reynolds numbers, are based on some type of modeling. This modeling can take a variety of forms:

1. Reynolds-stress models, which allow for the calculation of one-point first and second moments such as the mean velocity, mean pressure, and turbulent kinetic energy;
2. subgrid-scale models for large-eddy simulations, wherein the large, energy-containing eddies are computed directly and the effect of the small scales—which are more universal in character—are modeled;
3. two-point closures or spectral models, which provide more detailed information about the turbulence structure, since they are based on the two-point velocity correlation tensor; or

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4. pdf models based on the joint probability density function.

Large-eddy simulations (LES) have found several important geophysical applications in weather forecasting and in other atmospheric studies (cf Deardorff 1973, Clark & Farley 1984, Smolarkiewicz & Clark 1985). Likewise, LES has shed new light on the physics of certain basic turbulent flows—which include homogeneous shear flow and channel flow—at higher Reynolds numbers that are not accessible to direct simulations (cf Moin & Kim 1982, Bardina et al 1983, Rogallo & Moin 1984, Piomelli et al 1987). Two-point closures such as the EDQNM (Eddy-Damped Quasi-Normal Markovian) model of Orszag (1970) have been quite useful in the analysis of homogeneous turbulent flows, where they have provided new information on the structure of isotropic turbulence (cf Lesieur 1987) and on the effect of shear and rotation (cf Bertoglio 1982). However, there are a variety of theoretical and operational problems with two-point closures and large-eddy simulations that make their application to strongly inhomogeneous turbulent flows difficult, if not impossible—especially in irregular geometries with solid boundaries. There have been no applications of two-point closures to wall-bounded turbulent flows, and virtually all such applications of LES have been in simple geometries where Van Driest damping could be used—an empirical approach that generally does not work well when there is flow separation. Comparable problems in dealing with wall-bounded flows have, for the most part, limited pdf methods to free turbulent flows, where they have been quite useful in the description of chemically reacting turbulence (see Pope 1985). Since most practical engineering flows involve complex geometries with solid boundaries—at Reynolds numbers that are far higher than those that are accessible to direct simulations—the preferred approach has been to base such calculations on Reynolds-stress modeling.[1] This forms the motivation for the present review paper, whose purpose is to put into perspective some of the more recent theoretical developments in Reynolds-stress modeling.

The concept of Reynolds averaging was introduced by Sir Osborne Reynolds in his landmark turbulence research of the latter part of the nineteenth century (see Reynolds 1895). During a comparable time frame, Boussinesq (1877) introduced the concept of the turbulent or eddy viscosity as the basis for a simple time-averaged turbulence closure. However, it was not until after 1920 that the first successful calculation of a practical turbulent flow was achieved based on the Reynolds-averaged Navier-Stokes equations with an eddy-viscosity model. This was largely due to

[1] In fact, the only alternative of comparable simplicity is the vorticity transport theory of Taylor (1915); a three-dimensional vorticity covariance closure along these lines has been recently pursued by Bernard and coworkers (cf Bernard & Berger 1982).

the pioneering work of Prandtl (1925), who introduced the concept of the mixing length as a basis for the determination of the eddy viscosity. This mixing-length model led to closed-form solutions for turbulent pipe and channel flows that were remarkably successful in collapsing the existing experimental data. A variety of turbulence researchers—most notably von Kármán (1930, 1948)—made further contributions to the mixing-length approach, which continued to be a highly active area of research until the post–World War II period. By this time it was clear that the basic assumptions behind the mixing-length approach—which makes a direct analogy between turbulent transport processes and molecular transport processes—were unrealistic; turbulent flows do not have a clear-cut separation of scales. With the aim of developing more general models, Prandtl (1945) tied the eddy viscosity to the turbulent kinetic energy, which was obtained from a separate modeled transport equation. This was a precursor to the one-equation models of turbulence—or the so-called K-l models—wherein the turbulent length scale l is specified empirically and the turbulent kinetic energy K is obtained from a modeled transport equation. However, these models still suffered from the deficiencies intrinsic to all eddy-viscosity models: the inability to properly account for streamline curvature, body forces, and history effects on the individual Reynolds-stress components.

In a landmark paper by Rotta (1951), the foundation was laid for a full Reynolds-stress turbulence closure, which was to ultimately change the course of Reynolds-stress modeling. This new approach of Rotta—which is now referred to as second-order or second-moment closure—was based on the Reynolds-stress transport equation. By making use of some of the statistical ideas of A. M. Kolmogorov from the 1940s (and by introducing some entirely new ideas), Rotta succeeded in closing the Reynolds-stress transport equation. This new Reynolds-stress closure, unlike eddy-viscosity models, accounted for both history and nonlocal effects on the evolution of the Reynolds-stress tensor—features whose importance had long been known. However, since this approach required the solution of six additional transport equations for the individual components of the Reynolds-stress tensor, it was not computationally feasible for the next few decades to solve complex engineering flows based on a full second-order closure. By the 1970s, with the wide availability of high-speed computers, a new thrust in the development and implementation of second-order closure models began with the work of Daly & Harlow (1970) and Donaldson (1972). In an important paper, Launder, Reece & Rodi (1975) developed a new second-order closure model that improved significantly on the earlier work of Rotta (1951). More systematic models for the pressure-strain correlation and turbulent transport terms were derived by

Launder, Reece & Rodi; a modeled transport equation for the turbulent dissipation rate was also solved in conjunction with this Reynolds-stress model. However, more importantly, Launder, Reece & Rodi (1975) showed how second-order closure models could be calibrated and applied to the solution of practical turbulent flows. When the Launder, Reece & Rodi (1975) model is contracted and supplemented with an eddy-viscosity representation for the Reynolds stress, a two-equation model (referred to as the K-ε model) is obtained, which is almost identical to that derived by Hanjalić & Launder (1972) a few years earlier. Because of the substantially lower computational effort required, the K-ε model is still one of the most commonly used turbulence models for the solution of practical engineering problems.

Subsequent to the publication of the paper by Launder, Reece & Rodi (1975), various turbulence modelers have continued research on second-order closures. For example, Lumley (1978) implemented the important constraint of realizability and made significant contributions to the modeling of the pressure-strain correlation and buoyancy effects. Launder and coworkers continued to expand on the refinement and application of second-order closure models to problems of significant engineering interest (see Launder 1990). Speziale (1985, 1987a) exploited invariance arguments—along with consistency conditions for solutions of the Navier-Stokes equations in a rapidly rotating frame—to develop new models for the rapid pressure-strain correlation. Haworth & Pope (1986) developed a second-order closure model starting from the pdf-based Langevin equation. W. C. Reynolds (private communication, 1988) has attempted to develop models for the rapid pressure-strain correlation by using Rapid-Distortion Theory (RDT).

In this paper, analytical methods for the derivation of Reynolds-stress models are reviewed. Zero-, one-, and two-equation models are considered along with second-order closures. Two approaches to the development of models are discussed.

1. *The continuum mechanics approach*, which is typically based on a Taylor expansion. Invariance constraints—as well as other consistency conditions such as RDT and realizability—are then used to simplify the model. The remaining constants are evaluated by reference to benchmark physical experiments.
2. *The statistical mechanics approach*, which is based on the construction of an asymptotic expansion. Unlike in the continuum mechanics approach, here the constants of the model are calculated explicitly. The two primary examples of this approach are the two-scale Direct Interaction Approximation (DIA) models of Yoshizawa (1984) and the Renormalization Group (RNG) models of Yakhot & Orszag (1986).

The basic methodology of these two techniques are examined here; however, more emphasis is placed on the continuum-mechanics approach, since there is a larger body of literature on this method and since it has been the author's preferred approach. The strengths and weaknesses of a variety of Reynolds-stress models are discussed in detail and illustrated by examples. A strong case is made for the superior predictive capabilities of second-order closures in comparison to the older zero-, one-, and two-equation models. However, some significant deficiencies in the structure of second-order closures that still remain are pointed out. These issues, as well as the author's views concerning possible future directions of research, are discussed in the sections to follow.

BASIC EQUATIONS OF REYNOLDS-STRESS MODELING

We consider here the turbulent flow of a viscous, incompressible fluid with constant properties. (Limitations of space do not allow us to discuss compressible turbulence modeling in any detail.) The governing field equations are the Navier-Stokes and continuity equations, which are given by

$$\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}=-\frac{\partial p}{\partial x_i}+\nu\nabla^2 u_i, \tag{1}$$

$$\frac{\partial u_i}{\partial x_i}=0, \tag{2}$$

where u_i is the velocity vector, p is the modified pressure (which can include a gravitational potential), and ν is the kinematic viscosity of the fluid. In (1)–(2), the Einstein summation convention applies to repeated indices.

The velocity and pressure are decomposed into mean and fluctuating parts as follows:

$$u_i=\bar{u}_i+u'_i, \qquad p=\bar{p}+p'. \tag{3}$$

It is assumed that any flow variables ϕ and ψ obey the Reynolds-averaging rules (cf Tennekes & Lumley 1972):

$$\overline{\phi'}=\overline{\psi'}=0, \tag{4}$$

$$\overline{\phi\psi}=\bar{\phi}\,\bar{\psi}+\overline{\phi'\psi'}, \tag{5}$$

$$\overline{\phi'\bar{\psi}}=\overline{\psi'\bar{\phi}}=0. \tag{6}$$

In a statistically steady turbulence, the mean of a flow variable ϕ can be taken to be the simple time average

$$\bar{\phi} = \bar{\phi}^{(T)}(\mathbf{x}) \equiv \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \phi(\mathbf{x}, t)\, dt, \tag{7}$$

whereas for a spatially homogeneous turbulence, a volume average can be used, i.e.

$$\bar{\phi} = \bar{\phi}^{(V)}(t) \equiv \lim_{V\to\infty} \frac{1}{V}\int_{V} \phi(\mathbf{x}, t)\, d^3x. \tag{8}$$

For more general turbulent flows that are neither statistically steady nor homogeneous, the mean of any flow variable ϕ is taken to be the ensemble mean

$$\bar{\phi} = \bar{\phi}^{(E)}(\mathbf{x}, t) \equiv \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \phi^{(k)}(\mathbf{x}, t), \tag{9}$$

where an average is taken over N repeated experiments. The ergodic hypothesis is assumed to apply—namely, that in a statistically steady turbulent flow, it is assumed that

$$\bar{\phi}^{(T)} = \bar{\phi}^{(E)}, \tag{10}$$

and in a homogeneous turbulent flow it is assumed that

$$\bar{\phi}^{(V)} = \bar{\phi}^{(E)}. \tag{11}$$

The Reynolds-averaged Navier-Stokes equation—which physically corresponds to a balance of mean linear momentum—takes the form

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{\partial \bar{p}}{\partial x_i} + \nu \nabla^2 \bar{u}_i - \frac{\partial \tau_{ij}}{\partial x_j}, \tag{12}$$

where

$$\tau_{ij} = \overline{u_i' u_j'} \tag{13}$$

is the Reynolds-stress tensor. Equation (12) is obtained by substituting the decompositions (3) into the Navier-Stokes equation (1) and then taking an ensemble mean. The mean continuity equation is given by

$$\frac{\partial \bar{u}_i}{\partial x_i} = 0 \tag{14}$$

and is obtained by simply taking the ensemble mean of (2). Equations (12)–(14) do not represent a closed system for the determination of the mean velocity $\bar{u}_i$ and mean pressure $\bar{p}$ because of the additional six unknowns contained within the Reynolds-stress tensor. The problem of

Reynolds-stress closure is to tie the Reynolds-stress tensor to the mean-velocity field in some physically consistent fashion.

In order to gain greater insight into the problem of Reynolds-stress closure, we now consider the governing field equations for the turbulent fluctuations. The fluctuating momentum equation—from which u_i' is determined—takes the form

$$\frac{\partial u_i'}{\partial t}+\bar{u}_j\frac{\partial u_i'}{\partial x_j} = -u_j'\frac{\partial u_i'}{\partial x_j}-u_j'\frac{\partial \bar{u}_i}{\partial x_j}-\frac{\partial p'}{\partial x_i}+\nu\nabla^2 u_i'+\frac{\partial \tau_{ij}}{\partial x_j} \tag{15}$$

and is obtained by subtracting (12) from (1) after the decompositions (3) are introduced. The fluctuating continuity equation, which is obtained by subtracting (14) from (2), is given by

$$\frac{\partial u_i'}{\partial x_i} = 0. \tag{16}$$

Equations (15)–(16) have solutions for the fluctuating velocity u_i' that are of the general mathematical form

$$u_i'(\mathbf{x}, t) = \mathscr{F}_i[\bar{\mathbf{u}}(\mathbf{y}, s), \mathbf{u}'(\mathbf{y}, 0), \mathbf{u}'(\mathbf{y}, s)|_{\partial\mathscr{V}}; \mathbf{x}, t], \qquad \mathbf{y}\in\mathscr{V}, \quad s\in(-\infty, t), \tag{17}$$

where $\mathscr{F}_i[\cdot]$ denotes a functional, $\mathscr{V}$ is the volume of the fluid, and $\partial\mathscr{V}$ is its bounding surface. In alternative terms, the fluctuating velocity is a *functional of the global history of the mean-velocity field* with an implicit dependence on its own initial and boundary conditions. Here we use the term functional in its broadest mathematical sense—namely, any quantity determined by a function. From (17), we can explicitly calculate the Reynolds-stress tensor $\tau_{ij} \equiv \overline{u_i'u_j'}$, which will also be a functional of the global history of the mean velocity. However, there is a serious problem in regard to the dependence of τ_{ij} on the initial and boundary conditions for the fluctuating velocity, as discussed by Lumley (1970). There is no hope for a workable Reynolds-stress closure if there is a detailed dependence on such initial and boundary conditions. For turbulent flows that are sufficiently far from solid boundaries—and sufficiently far evolved in time past their initiation—it is not unreasonable to assume that the initial and boundary conditions on the fluctuating velocity (beyond those for τ_{ij}) merely set the length and time scales of the turbulence. Hence, with this crucial assumption, we obtain the expression

$$\tau_{ij}(\mathbf{x}, t) = \mathscr{F}_{ij}[\bar{\mathbf{u}}(\mathbf{y}, s), l_0(\mathbf{y}, s), \tau_0(\mathbf{y}, s); \mathbf{x}, t], \qquad \mathbf{y}\in\mathscr{V}, \quad s\in(-\infty, t), \tag{18}$$

where l_0 is the turbulent length scale, τ_0 is the turbulent time scale, and the functional $\mathscr{F}_{ij}$ depends implicitly on the initial and boundary conditions for τ_{ij} [see Lumley (1970) for a more detailed discussion of these points]. Equation (18) serves as the cornerstone of Reynolds-stress modeling. Eddy-viscosity models, which are of the form

$$\tau_{ij} = -\nu_{\mathrm{T}}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{19}$$

(where the turbulent or eddy viscosity $\nu_{\mathrm{T}} \propto l_0^2/\tau_0$), represent one of the simplest examples of (18). Of course, the assumption that the Reynolds-stress tensor can be characterized by a *single* length and time scale constitutes an idealization. Turbulent flows exhibit a wide range of excited length and time scales; this is precisely the reason that they are so difficult to compute directly.

Since we discuss second-order closure models later, it is useful at this point to introduce the Reynolds-stress transport equation as well as the turbulent dissipation-rate transport equation. The latter equation plays an important role in many commonly used Reynolds-stress models where the turbulent dissipation rate is used to build up the turbulent length and time scales. If we denote the fluctuating momentum equation (15) in operator form as

$$\mathscr{L}u_i' = 0, \tag{20}$$

then the Reynolds-stress transport equation is obtained from the second moment

$$\overline{u_i'\mathscr{L}u_j' + u_j'\mathscr{L}u_i'} = 0, \tag{21}$$

whereas the turbulent dissipation rate is obtained from the moment

$$2\nu\overline{\frac{\partial u_i'}{\partial x_j}\frac{\partial}{\partial x_j}(\mathscr{L}u_i')} = 0. \tag{22}$$

More explicitly, the Reynolds-stress transport equation (21) is given by (cf Hinze 1975)

$$\frac{\partial \tau_{ij}}{\partial t} + \bar{u}_k\frac{\partial \tau_{ij}}{\partial x_k} = -\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} - \frac{\partial C_{ijk}}{\partial x_k} + \nu\nabla^2\tau_{ij}, \tag{23}$$

where

$$\Pi_{ij} \equiv \overline{p'\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)}, \tag{24}$$

$$\varepsilon_{ij} = 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_j'}{\partial x_k}}, \tag{25}$$

$$C_{ijk} \equiv \overline{u_i' u_j' u_k'} + \overline{p' u_i'}\delta_{jk} + \overline{p' u_j'}\delta_{ik} \tag{26}$$

are the pressure-strain correlation, dissipation-rate correlation, and third-order diffusion correlation, respectively. On the other hand, the turbulent dissipation rate transport equation (22) is given by

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = \nu \nabla^2 \varepsilon - 2\nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_j}} \frac{\partial \bar{u}_i}{\partial x_k} - 2\nu \overline{\frac{\partial u_j'}{\partial x_i} \frac{\partial u_j'}{\partial x_k}} \frac{\partial \bar{u}_i}{\partial x_k} - 2\nu \overline{u_k' \frac{\partial u_i'}{\partial x_j}} \frac{\partial^2 \bar{u}_i}{\partial x_k \partial x_j}$$
$$- 2\nu \overline{\frac{\partial u_i'}{\partial x_k} \frac{\partial u_i'}{\partial x_m} \frac{\partial u_k'}{\partial x_m}} - \nu \frac{\partial}{\partial x_k} \left(\overline{u_k' \frac{\partial u_i'}{\partial x_m} \frac{\partial u_i'}{\partial x_m}} \right) - 2\nu \frac{\partial}{\partial x_k} \left(\overline{\frac{\partial p'}{\partial x_m} \frac{\partial u_k'}{\partial x_m}} \right)$$
$$- 2\nu^2 \overline{\frac{\partial^2 u_i'}{\partial x_k \partial x_m} \frac{\partial^2 u_i'}{\partial x_k \partial x_m}}, \tag{27}$$

where $\varepsilon \equiv \frac{1}{2}\varepsilon_{ii}$ is the scalar dissipation rate. The seven higher order correlations on the right-hand side of (27) correspond to three physical effects: The first four terms give rise to the production of dissipation, the next two terms represent the turbulent diffusion of dissipation, and the last term represents the turbulent destruction of dissipation.

Finally, before closing this section, it would be useful to briefly discuss two constraints that have played a central role in the formulation of modern Reynolds-stress models: realizability and frame invariance. The constraint of realizability was first posed by Schumann (1977) and then rigorously introduced by Lumley into Reynolds-stress transport models [see Lumley (1978, 1983) for a more detailed discussion]. It requires that a Reynolds-stress model yield positive component energies, i.e. that

$$\tau_{\alpha\alpha} \geq 0, \qquad \alpha = 1, 2, 3 \tag{28}$$

for any turbulent flow. The inequality (28) (where Greek indices are used to indicate that there is no summation) is a direct consequence of the definition of the Reynolds-stress tensor given by (13). It was first shown by Lumley that realizability could be satisfied *identically* in homogeneous turbulent flows by Reynolds-stress transport models; this is accomplished by requiring that whenever a component energy $\tau_{\alpha\alpha}$ vanishes, its time rate $\dot{\tau}_{\alpha\alpha}$ also vanishes.

Donaldson was probably the first to advocate the unequivocal use of coordinate invariance in turbulence modeling (cf Donaldson & Rosenbaum 1968). This approach, which Donaldson termed "invariant modeling," was based on the Reynolds-stress transport equation and required that all modeled terms be cast in tensor form. Prior to the 1970s it was not uncommon for turbulence models to be proposed that were incapable of being uniquely put in tensor form. (Hence, these older models could not be properly extended to more complex flows, particularly to ones involving curvilinear coordinates.) The more complicated question of frame invariance—where *time-dependent* rotations and translations of the reference frame are accounted for—was first considered by Lumley (1970) in an interesting paper. A more comprehensive analysis of the effect of a change of reference frame was conducted by the present author in a series of papers published during the 1980s [see Speziale (1989) for a detailed review of these results]. In a general noninertial reference frame, which can undergo *arbitrary time-dependent* rotations and translations relative to an inertial frame, the fluctuating momentum equation takes the form

$$\frac{\partial u_i'}{\partial t}+\bar{u}_j\frac{\partial u_i'}{\partial x_j} = -u_j'\frac{\partial u_i'}{\partial x_j}-u_j'\frac{\partial \bar{u}_i}{\partial x_j}-\frac{\partial p'}{\partial x_i}+\nu\nabla^2 u_i'+\frac{\partial \tau_{ij}}{\partial x_j}-2e_{ijk}\Omega_j u_k', \tag{29}$$

where e_{ijk} is the permutation tensor, and Ω_i is the rotation rate of the reference frame relative to an inertial framing (see Speziale 1989). From (29), it is clear that the evolution of the fluctuating velocity only depends directly on the motion of the reference frame through the Coriolis acceleration; translational accelerations—as well as centrifugal and angular accelerations—only have an indirect effect through the changes that they induce in the mean-velocity field. Consequently, closure models for the Reynolds-stress tensor must be form invariant under the *extended* Galilean group of transformations

$$\mathbf{x}^* = \mathbf{x}+\mathbf{c}(t), \tag{30}$$

which allows for an arbitrary *translational acceleration* $-\ddot{\mathbf{c}}$ of the reference frame relative to an inertial framing $\mathbf{x}$.

In the limit of two-dimensional turbulence (or a turbulence where the ratio of the fluctuating to mean time scales $\tau_0/T_0 \ll 1$), the Coriolis acceleration is derivable from a scalar potential that can be absorbed into the fluctuating pressure (or neglected), yielding complete frame indifference (see Speziale 1981, 1983). This invariance under arbitrary time-dependent rotations and translations of the reference frame specified by

$$\mathbf{x}^* = \mathbf{Q}(t)\mathbf{x}+\mathbf{c}(t) \tag{31}$$

[where $\mathbf{Q}(t)$ is any *time-dependent* proper-orthogonal rotation tensor] is referred to as Material Frame Indifference (MFI), the term that has been traditionally used for the analogous manifest invariance of constitutive equations in modern continuum mechanics. For general three-dimensional turbulent flows where $\tau_0/T_0 = O(1)$, MFI does not apply as a result of Coriolis effects, as was first pointed out by Lumley (1970). However, the Coriolis acceleration in (29) can be combined with the mean velocity in such a way that frame dependence enters *exclusively* through the appearance of the intrinsic or absolute mean-vorticity tensor defined by (see Speziale 1989)

$$\overline{W}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right) + e_{mji}\Omega_m. \tag{32}$$

This result, along with the constraint of MFI in the two-dimensional limit, restricts the allowable form of models considerably.

ZERO-EQUATION AND ONE-EQUATION MODELS BASED ON AN EDDY VISCOSITY

In the simplest continuum mechanics approach—whose earliest formulations have often been referred to as phenomenological models—the starting point is Equation (18). Invariance under the extended Galilean group of transformations (30)—which any physically sound Reynolds-stress model must obey—can be satisfied identically by models of the form

$$\tau_{ij}(\mathbf{x}, t) = \mathscr{F}_{ij}[\bar{\mathbf{u}}(\mathbf{y}, s) - \bar{\mathbf{u}}(\mathbf{x}, s), l_0(\mathbf{y}, s); \tau_0(\mathbf{y}, s); \mathbf{x}, t],$$
$$\mathbf{y} \in \mathscr{V}, \quad s \in (-\infty, t). \tag{33}$$

The variables $\bar{\mathbf{u}}(\mathbf{y}, s) - \bar{\mathbf{u}}(\mathbf{x}, s)$, $l_0(\mathbf{y}, s)$, and $\tau_0(\mathbf{y}, s)$ can be expanded in a Taylor series as follows:

$$\bar{\mathbf{u}}(\mathbf{y}, s) - \bar{\mathbf{u}}(\mathbf{x}, s) = (y_i - x_i)\frac{\partial \bar{\mathbf{u}}}{\partial x_i} + \frac{(y_i - x_i)(y_j - x_j)}{2!}\frac{\partial^2 \bar{\mathbf{u}}}{\partial x_i \partial x_j} + (s-t)(y_i - x_i)\frac{\partial^2 \bar{\mathbf{u}}}{\partial t \partial x_i} + \cdots, \tag{34}$$

$$l_0(\mathbf{y}, s) = l_0 + (y_i - x_i)\frac{\partial l_0}{\partial x_i} + (s-t)\frac{\partial l_0}{\partial t} + \frac{(s-t)^2}{2!}\frac{\partial^2 l_0}{\partial t^2} + \frac{(y_i - x_i)(y_j - x_j)}{2!}\frac{\partial^2 l_0}{\partial x_i \partial x_j} + (s-t)(y_i - x_i)\frac{\partial^2 l_0}{\partial t \partial x_i} + \cdots, \tag{35}$$

$$\tau_0(\mathbf{y}, s) = \tau_0 + (y_i - x_i)\frac{\partial \tau_0}{\partial x_i} + (s-t)\frac{\partial \tau_0}{\partial t} + \frac{(s-t)^2}{2!}\frac{\partial^2 \tau_0}{\partial t^2} + \frac{(y_i - x_i)(y_j - x_j)}{2!}\frac{\partial^2 \tau_0}{\partial x_i \partial x_j} + (s-t)(y_i - x_i)\frac{\partial^2 \tau_0}{\partial t \partial x_i} + \cdots, \tag{36}$$

where terms up to the second order are shown and it is understood that $\bar{\mathbf{u}}$, l_0, and τ_0 on the right-hand side of (34)–(36) are evaluated at $\mathbf{x}$ and t. After splitting τ_{ij} into isotropic and deviatoric parts—and applying elementary dimensional analysis—the following expression is obtained:

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \frac{l_0^2}{\tau_0^2}\hat{\mathscr{F}}_{ij}[\bar{\mathbf{v}}(\mathbf{y}, s) - \bar{\mathbf{v}}(\mathbf{x}, s); \mathbf{x}, t], \qquad \mathbf{y} \in \mathscr{V}, \quad s \in (-\infty, t), \tag{37}$$

where

$$\bar{\mathbf{v}} = \frac{\tau_0 \bar{\mathbf{u}}}{l_0}, \qquad K = \frac{1}{2}\tau_{ii} \tag{38}$$

are, respectively, the dimensionless mean velocity and the turbulent kinetic energy. $\hat{\mathscr{F}}_{ij}$ is a traceless and dimensionless functional of its arguments. By making use of the Taylor expansions (34)—(36), it is a simple matter to show that

$$\bar{v}_i(\mathbf{y}, s) - \bar{v}_i(\mathbf{x}, s) = \frac{\tau_0}{T_0}(y_j^* - x_j^*)\left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^* + O\left(\frac{\tau_0^2}{T_0^2}\right), \tag{39}$$

where

$$y_i^* - x_i^* \equiv \frac{y_i - x_i}{l_0}, \qquad \left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^* \equiv T_0 \frac{\partial \bar{u}_i}{\partial x_j} \tag{40}$$

are dimensionless variables of order one, given that T_0 is the time scale of the mean flow. If, analogous to the molecular fluctuations of most continuum flows, we assume that there is a complete separation of scales such that

$$\frac{\tau_0}{T_0} \ll 1, \qquad \frac{l_0}{L_0} \ll 1, \tag{41}$$

Equation (37) can then be localized in space and time. Of course, it is well known that this constitutes an oversimplification; the molecular fluctuations of most continuum flows are such that $\tau_0/T_0 \leq 10^{-6}$, whereas with turbulent fluctuations, τ_0/T_0 can be of $O(1)$.

By making use of (39)–(41), Equation (37) can be localized to the approximate form

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \frac{l_0^2}{\tau_0^2}G_{ij}(\bar{v}_{k,l}), \tag{42}$$

where

$$\bar{v}_{k,l} \equiv \frac{\tau_0}{T_0}\left(\frac{\partial \bar{u}_k}{\partial x_l}\right)^* \tag{43}$$

is the dimensionless mean-velocity gradient. Since the tensor function G_{ij} is symmetric and traceless (and since $\bar{v}_{i,j}$ is traceless), it follows that, to the first order in τ_0/T_0, form invariance under a change of coordinates simplifies (42) to (cf Smith 1971)

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \nu_{\mathrm{T}}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{44}$$

where

$$\nu_{\mathrm{T}} \equiv l_0^2/\tau_0 \tag{45}$$

is the eddy viscosity. While the standard eddy-viscosity model (44) comes out of this derivation when only first-order terms in τ_0/T_0 are maintained, anisotropic eddy-viscosity (or viscoelastic) models are obtained when second-order terms are maintained. These more complicated models are discussed in the next section.

Eddy-viscosity models are not closed until prescriptions are made for the turbulent length and time scales in (45). In zero-equation models, both l_0 and τ_0 are prescribed algebraically. The earliest example of a successful zero-equation model is Prandtl's mixing-length theory (see Prandtl 1925). By making analogies between the turbulent length scale and the mean free path in the kinetic theory of gases, Prandtl argued that ν_{T} should be of the form

$$\nu_{\mathrm{T}} = l_m^2\left|\frac{d\bar{u}}{dy}\right| \tag{46}$$

for a plane shear flow where the mean velocity is of the form $\bar{\mathbf{u}} = \bar{u}(y)\mathbf{i}$. In (46), l_m is the "mixing length," which represents the distance traversed by a small lump of fluid before losing its momentum. Near a plane solid boundary, it was furthermore assumed that

$$l_m = \kappa y, \tag{47}$$

where κ is the von Kármán constant. (This result can be obtained from a first-order Taylor-series expansion, since l_m must vanish at a wall.) When

(46)–(47) are used in conjunction with the added assumption that the shear stress is approximately constant in the near-wall region, the celebrated "law of the wall" is obtained:

$$u^+ = \frac{1}{\kappa}\ln y^+ + C, \tag{48}$$

where y^+ is measured normal from the wall and

$$u^+ \equiv \frac{\bar{u}}{u_\tau}, \qquad y^+ \equiv \frac{yu_\tau}{\nu}. \tag{49}$$

Here u_τ is the friction velocity, and C is a dimensionless constant. Equation (48) (with $\kappa \doteq 0.4$ and $C \doteq 5.0$) was remarkably successful in collapsing the experimental data for turbulent pipe and channel flows for a significant range of y^+ varying from 30 to 1000 [see Schlichting (1968) for an interesting review of these results]. The law of the wall is still heavily used to this day as a boundary condition in the more sophisticated turbulence models for which it is either difficult or too computationally expensive to integrate directly to a solid boundary.

During the 1960s and 1970s, with the dramatic emergence of computational fluid dynamics, some efforts were made to generalize mixing-length models to three-dimensional turbulent flows. With such models, Reynolds-averaged computations could be conducted with any existing Navier-Stokes computer code that allowed for a variable viscosity. Prandtl's mixing-length theory (46) has two straightforward extensions to three dimensional flows: the strain-rate form

$$\nu_T = l_m^2(2\bar{S}_{ij}\bar{S}_{ij})^{1/2}, \tag{50}$$

where $\bar{S}_{ij} \equiv \frac{1}{2}(\partial\bar{u}_i/\partial x_j + \partial\bar{u}_j/\partial x_i)$ is the mean rate-of-strain tensor; or the vorticity form

$$\nu_T = l_m^2(\bar{\omega}_i\bar{\omega}_i)^{1/2}, \tag{51}$$

where $\bar{\omega}_i = e_{ijk}\partial\bar{u}_k/\partial x_j$ is the mean-vorticity vector. The former model (50) is due to Smagorinsky (1963) and has been primarily used as a subgrid-scale model for large-eddy simulations; the latter model (51) is due to Baldwin & Lomax (1978) and has been widely used for Reynolds-averaged aerodynamic computations. Both models—which collapse to Prandtl's mixing-length theory (46) in a plane shear flow—have the primary advantage of their computational ease of application. They suffer from the disadvantages of the need for an ad hoc prescription of the turbulent length scale in each problem solved and of the complete neglect of history effects. Furthermore, they do not provide for the computation of the turbulent kinetic energy, which is a crucial measure of the intensity of the turbulence fluctuations. (Such zero-equation models only allow for the calculation of the mean velocity and mean pressure.)

One-equation models were developed in order to eliminate some of the deficiencies cited above—namely, to provide for the computation of the turbulent kinetic energy and to account for some limited nonlocal and history effects in the determination of the eddy viscosity. In these one-equation models of turbulence, the eddy viscosity is assumed to be of the form (see Kolmogorov 1942, Prandtl 1945)

$$\nu_{\mathrm{T}} = K^{1/2} l, \tag{52}$$

where the turbulent kinetic energy K is obtained from a modeled version of its exact transport equation

$$\frac{\partial K}{\partial t} + \bar{u}_i \frac{\partial K}{\partial x_i} = -\tau_{ij} \frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon - \frac{\partial}{\partial x_i}\left(\frac{1}{2}\overline{u_k' u_k' u_i'} + \overline{p' u_i'}\right) + \nu \nabla^2 K. \tag{53}$$

Equation (53), which is obtained by a simple contraction of (23), can be closed once models for the turbulent transport and dissipation terms [i.e. the second and third terms on the right-hand side of (53)] are provided. Consistent with the assumption that there is a clear-cut separation of scales (i.e. that the turbulent transport processes parallel the molecular ones), the turbulent transport term is modeled by a gradient transport hypothesis,

$$\frac{1}{2}\overline{u_k' u_k' u_i'} + \overline{p' u_i'} = -\frac{\nu_{\mathrm{T}}}{\sigma_{\mathrm{K}}} \frac{\partial K}{\partial x_i}, \tag{54}$$

where σ_{K} is a dimensionless constant. By simple scaling arguments—analogous to those made by Kolmogorov (1942) for high-turbulence Reynolds numbers—the turbulent dissipation rate ε is usually modeled as follows:

$$\varepsilon = C^* \frac{K^{3/2}}{l}, \tag{55}$$

where C^* is a dimensionless constant. A closed system of equations for the determination of $\bar{u}_i$, $\bar{p}$, and K is obtained once the turbulent length scale l is specified empirically. It should be mentioned that the modeled transport equation for the turbulent kinetic energy specified by Equations (53)–(55) cannot be integrated to a solid boundary. Either wall functions must be used or low-Reynolds-number versions of (53)–(55) must be substituted (cf Norris & Reynolds 1975, Reynolds 1976). It is interesting to note that Bradshaw et al (1967) considered an alternative one-equation model, based on a modeled transport equation for the Reynolds shear stress $\overline{u'v'}$, which seemed to be better suited for turbulent boundary layers.

Since zero- and one-equation models have not been in the forefront of turbulence-modeling research for the past 20 years, we do not present the results of any illustrative calculations. [The reader is referred to Cebeci & Smith (1974), Rodi (1980), and Bradshaw et al (1981) for some interesting examples.] The primary deficiencies of these models are twofold: (*a*) the use of an eddy viscosity, and (*b*) the need to provide an ad hoc specification of the turbulence length scale. This latter deficiency with regard to the length scale makes zero- and one-equation models incomplete; the two-equation models that are discussed in the next section were the first complete turbulence models (i.e. models that only require the specification of initial and boundary conditions for the solution of problems). Nonetheless, despite these deficiencies, zero- and one-equation models have made some important contributions to the computation of practical engineering flows. Their simplicity of structure—and reduced computing times—continue to make them the most commonly adopted models for complex aerodynamic calculations [see Cebeci & Smith (1968) and Johnson & King (1984) for two of the most popular such models].

TWO-EQUATION MODELS

A variety of two-equation models—which are among the most popular Reynolds-stress models for scientific and engineering calculations—are discussed in this section. Models of the K-ε, K-l, and K-ω type are considered based on an isotropic as well as an anisotropic eddy viscosity. Both the continuum mechanics and statistical mechanics approach for deriving such two-equation models are discussed.

The feature that distinguishes two-equation models from zero- or one-equation models is that *two separate modeled transport equations* are solved for the turbulent length and time scales (or for any two linearly independent combinations thereof). In the standard K-ε model—which is probably the most popular such model—the length and time scales are built up from the turbulent kinetic energy and dissipation rate as follows (see Hanjalić & Launder 1972, Launder & Spalding 1974):

$$l_0 \propto \frac{K^{3/2}}{\varepsilon}, \qquad \tau_0 \propto \frac{K}{\varepsilon}.$$

Separate modeled transport equations are solved for the turbulent kinetic energy K and turbulent dissipation rate ε. In order to close the exact transport equation for K, only a model for the turbulent transport term on the right-hand side of (53) is needed; consistent with the overriding assumption that there is a clear-cut separation of scales, the gradient transport model (54) is used. The exact transport equation for ε, given by (27), can be rewritten in the form

$$\frac{\partial \varepsilon}{\partial t}+\bar{u}_i\frac{\partial \varepsilon}{\partial x_i} = \nu\nabla^2\varepsilon+\mathscr{P}_\varepsilon+\mathscr{D}_\varepsilon-\Phi_\varepsilon, \tag{56}$$

where $\mathscr{P}_\varepsilon$ represents the production of dissipation [given by the first four correlations on the right-hand side of (27)], $\mathscr{D}_\varepsilon$ represents the turbulent diffusion of dissipation [given by the next two correlations on the right-hand side of (27)], and Φ_ε represents the turbulent destruction of dissipation [given by the last term on the right-hand side of (27)]. Again, consistent with the underlying assumption (41), a gradient transport hypothesis is used to model $\mathscr{D}_\varepsilon$:

$$\mathscr{D}_\varepsilon = \frac{\partial}{\partial x_i}\left(\frac{\nu_\mathrm{T}}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right), \tag{57}$$

where σ_ε is a dimensionless constant. The production of dissipation and destruction of dissipation are modeled as follows:

$$\mathscr{P}_\varepsilon = \mathscr{P}_\varepsilon\left(b_{ij}, \frac{\partial \bar{u}_i}{\partial x_j}, K, \varepsilon\right), \tag{58}$$

$$\Phi_\varepsilon = \Phi_\varepsilon(K, \varepsilon), \tag{59}$$

where $b_{ij} \equiv (\tau_{ij}-\frac{2}{3}K\delta_{ij})/2K$ is the anisotropy tensor. Equations (58)–(59) are based on the physical reasoning that the production of dissipation is governed by the level of anisotropy in the Reynolds-stress tensor and the mean-velocity gradients (scaled by K and ε, which determine the length and time scales), whereas the destruction of dissipation is determined by the length and time scales alone (an assumption motivated by isotropic turbulence). By simple dimensional analysis it follows that

$$\Phi_\varepsilon = C_{\varepsilon 2}\frac{\varepsilon^2}{K}, \tag{60}$$

where $C_{\varepsilon 2}$ is a dimensionless constant. Coordinate invariance coupled with a simple dimensional analysis yields

$$\mathscr{P}_\varepsilon = -2C_{\varepsilon 1}\varepsilon b_{ij}\frac{\partial \bar{u}_i}{\partial x_j} \equiv -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} \tag{61}$$

as the leading term in a Taylor-series expansion of (58) under the assumption that $\|\mathbf{b}\|$ and τ_0/T_0 are small. ($C_{\varepsilon 1}$ is a dimensionless constant.) Equation (61) was originally postulated based on the simple physical reasoning that the production of dissipation should be proportional to the production of turbulent kinetic energy (cf Hanjalić & Launder 1972). A composition of these various modeled terms yields the standard K-ε model (cf Launder & Spalding 1974):

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - \nu_{\mathrm{T}}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \tag{62a}$$

$$\nu_{\mathrm{T}} = C_\mu \frac{K^2}{\varepsilon}, \tag{62b}$$

$$\frac{\partial K}{\partial t} + \bar{u}_i \frac{\partial K}{\partial x_i} = -\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - \varepsilon + \frac{\partial}{\partial x_i}\left(\frac{\nu_{\mathrm{T}}}{\sigma_{\mathrm{K}}}\frac{\partial K}{\partial x_i}\right) + \nu\nabla^2 K, \tag{62c}$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} + \frac{\partial}{\partial x_i}\left(\frac{\nu_{\mathrm{T}}}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right) + \nu\nabla^2\varepsilon. \tag{62d}$$

Here, the constants assume the approximate values of $C_\mu = 0.09$, $\sigma_{\mathrm{K}} = 1.0$, $\sigma_\varepsilon = 1.3$, $C_{\varepsilon 1} = 1.44$, and $C_{\varepsilon 2} = 1.92$, which are obtained (for the most part) by comparisons of the model predictions with the results of physical experiments on equilibrium turbulent boundary layers and the decay of isotropic turbulence. It should be noted that the standard K-ε model (62) cannot be integrated to a solid boundary; either wall functions or some form of damping must be implemented [see Patel et al (1985) for an extensive review of these methods].

At this point, it is instructive to provide some examples of the performance of the K-ε model in benchmark, homogeneous turbulent flows as well as in a nontrivial, inhomogeneous turbulent flow. It is a simple matter to show that in isotropic turbulence where

$$\tau_{ij} = \frac{2}{3}K(t)\delta_{ij}, \qquad \varepsilon_{ij} = \frac{2}{3}\varepsilon(t)\delta_{ij},$$

the K-ε model predicts the following rate of decay of the turbulent kinetic energy (cf Reynolds 1987):

$$K(t) = K_0[1 + (C_{\varepsilon 2} - 1)\varepsilon_0 t/K_0]^{-1/(C_{\varepsilon 2}-1)}. \tag{63}$$

Equation (63) indicates a power-law decay where $K \sim t^{-1.1}$—a result that is not far removed from what is observed in physical experiments (cf Comte-Bellot & Corrsin 1971).

Homogeneous shear flow constitutes another classical turbulent flow that has been widely used to evaluate models. In this flow, an initially

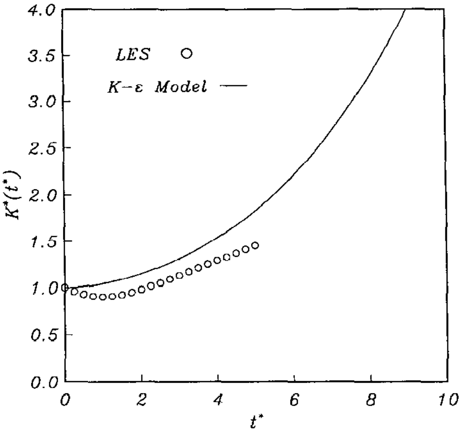

Figure 1 Time evolution of the turbulent kinetic energy in homogeneous shear flow. Comparison of the predictions of the standard K-ε model with the large-eddy simulation (LES) of Bardina et al (1983) for $\varepsilon_0/SK_0 = 0.296$.

isotropic turbulence is subjected to a constant shear rate S with mean-velocity gradients

$$\frac{\partial \bar{u}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{64}$$

The time evolution of the turbulent kinetic energy obtained from the standard K-ε model is compared in Figure 1 with the large-eddy simulation of Bardina et al (1983). (Here, $K^* \equiv K/K_0$ is the dimensionless kinetic energy, and $t^* \equiv St$ is the dimensionless time.) In so far as the equilibrium states are concerned, the standard K-ε model predicts (see Speziale & Mac Giolla Mhuiris 1989a) that $(b_{12})_\infty = -0.217$ and $(SK/\varepsilon)_\infty = 4.82$, in comparison to the experimental values of $(b_{12})_\infty = -0.15$ and $(SK/\varepsilon)_\infty = 6.08$, respectively[2] (see Tavoularis & Corrsin 1981). Consistent with a wide range of physical and numerical experiments, the standard K-ε model predicts that the equilibrium structure of homogeneous shear flow is *universal* (i.e. attracts all initial conditions) in the phase space of b_{ij} and SK/ε. Hence, from Figure 1 and the equilibrium results given above, it is clear that the K-ε model yields a qualitatively good description of shear

[2] Here, $(\cdot)_\infty$ denotes the equilibrium value obtained in the limit as $t \to \infty$.

flow; the specific quantitative predictions, however, could be improved upon.

As an example of the performance of the standard K-ε model in a more complicated inhomogeneous turbulence, the case of turbulent flow past a backward-facing step at a Reynolds number Re $\sim$ 100,000 is now presented. [This is the same test case as that considered at the 1980/81 AFOSR-HTTM-Stanford Conference on Complex Turbulent Flows; it corresponds to the experimental test case of Kim et al (1980).] In Figures 2*a*,*b* the mean-flow streamlines and turbulence-intensity profiles predicted by the K-ε model are compared with the experimental data of Kim et al (1980). The standard K-ε model—integrated using a single-layer log wall region starting at $y^+ = 30$—predicts a reattachment point of $x/\Delta H \doteq 5.7$ in comparison to the experimental mean value of $x/\Delta H \doteq 7.0$. This error, which is of the order of 20%, is comparable to that which occurs in the predicted turbulence intensities [see Figure 2*b* and Speziale & Ngo (1988) for more detailed comparisons]. However, Avva et al (1988) reported an improved prediction of $x/\Delta H \doteq 6.3$ for the reattachment point by using a fine mesh and a double-layer log wall region.

Recently, Yakhot & Orszag (1986) derived a K-ε model based on Renormalization Group (RNG) methods. In this approach, an expansion is made about an equilibrium state with known Gaussian statistics by making use of the correspondence principle. Bands of high wavenumbers (i.e. small scales) are systematically removed and space is rescaled. The dynamical equations for the renormalized (large-scale) velocity field account for the effect of the small scales that have been removed through the presence of an eddy viscosity. The removal of only the smallest scales gives rise to subgrid-scale models for large-eddy simulations; the removal of successively larger scales ultimately gives rise to Reynolds-stress models. In the high-Reynolds-number limit, the RNG-based K-ε model of Yakhot & Orszag (1986) is identical in form to the standard K-ε model (62). However, the constants of the model are *calculated explicitly* by the theory to be $C_\mu = 0.0837$, $C_{\varepsilon 1} = 1.063$, $C_{\varepsilon 2} = 1.7215$, $\sigma_K = 0.7179$, and $\sigma_\varepsilon = 0.7179$. Beyond having the attractive feature of no undetermined constants, the RNG K-ε model of Yakhot & Orszag (1986) automatically bridges the eddy viscosity to the molecular viscosity as a solid boundary is approached, eliminating the need for the use of empirical wall functions or Van Driest damping. It must be mentioned, however, that some problems with the specific numerical values of the constants in the RNG K-ε model have recently surfaced. In particular, the value of $C_{\varepsilon 1} = 1.063$ is dangerously close to $C_{\varepsilon 1} = 1$, which constitutes a singular point of the ε-transport equation. For example, the growth rate λ of the turbulent kinetic energy (where $K \sim e^{\lambda t^*}$ for $\lambda t^* \gg 1$) predicted by the K-ε model in homogeneous shear flow is given by (see Speziale & Mac Giolla Mhuiris 1989a)

$$\lambda = \left[\frac{C_\mu (C_{\varepsilon 2} - C_{\varepsilon 1})^2}{(C_{\varepsilon 1} - 1)(C_{\varepsilon 2} - 1)}\right]^{1/2}, \tag{65}$$

which becomes singular when $C_{\varepsilon 1} = 1$. Consequently, the value of $C_{\varepsilon 1} = 1.063$ derived by Yakhot & Orszag (1986) yields excessively large growth rates for the turbulent kinetic energy in homogeneous shear flow in comparison to both physical and numerical experiments (see Speziale et al 1989).

One of the major deficiencies of the standard K-ε model lies in its use of an eddy-viscosity model for the Reynolds-stress tensor. Eddy-viscosity models have two major problems associated with them: (*a*) They are purely dissipative and hence cannot account for Reynolds-stress relaxation effects, and (*b*) they are oblivious to the presence of rotational strains (e.g. they fail to distinguish between the physically distinct cases of plane shear, plane strain, and rotating plane shear). In order to overcome these deficiencies, a considerable research effort has been directed toward the development of nonlinear or anisotropic generalizations of eddy-viscosity models. By keeping second-order terms in the Taylor expansions (34)–(36), subject to invariance under the extended Galilean group (30), a more general representation for the Reynolds-stress tensor is obtained:

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - 2\frac{l_0^2}{\tau_0}\bar{S}_{ij} + \alpha_1 l_0^2\left(\bar{S}_{ik}\bar{S}_{kj} - \frac{1}{3}\bar{S}_{mn}\bar{S}_{mn}\delta_{ij}\right)$$
$$+\alpha_2 l_0^2\left(\bar{\omega}_{ik}\bar{\omega}_{kj} - \frac{1}{3}\bar{\omega}_{mn}\bar{\omega}_{mn}\delta_{ij}\right) + \alpha_3 l_0^2(\bar{S}_{ik}\bar{\omega}_{jk} + \bar{S}_{jk}\bar{\omega}_{ik})$$
$$+\alpha_4 l_0^2\left(\frac{\partial \bar{S}_{ij}}{\partial t} + \bar{\mathbf{u}}\cdot\nabla\bar{S}_{ij}\right), \tag{66}$$

where

$$\bar{S}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right), \qquad \bar{\omega}_{ij} = \frac{1}{2}\left(\frac{\partial \bar{u}_i}{\partial x_j} - \frac{\partial \bar{u}_j}{\partial x_i}\right) \tag{67}$$

are the mean rate-of-strain and mean-vorticity tensors [α_1, ..., α_4 are dimensionless constants; in the linear limit as $\alpha_i \to 0$, the eddy-viscosity model (44) is recovered]. When $\alpha_4 = 0$, the deviatoric part of (66) is of the general form $\tau_{ij} = A_{ijkl}\partial\bar{u}_k/\partial x_l$ (where A_{ijkl} depends algebraically on the mean-velocity gradients), and hence the term "anisotropic eddy-viscosity model" has been used in the literature. These models are probably more accurately characterized as "nonlinear" or "viscoelastic" corrections to the eddy-viscosity models. Lumley (1970) was probably the first to system-

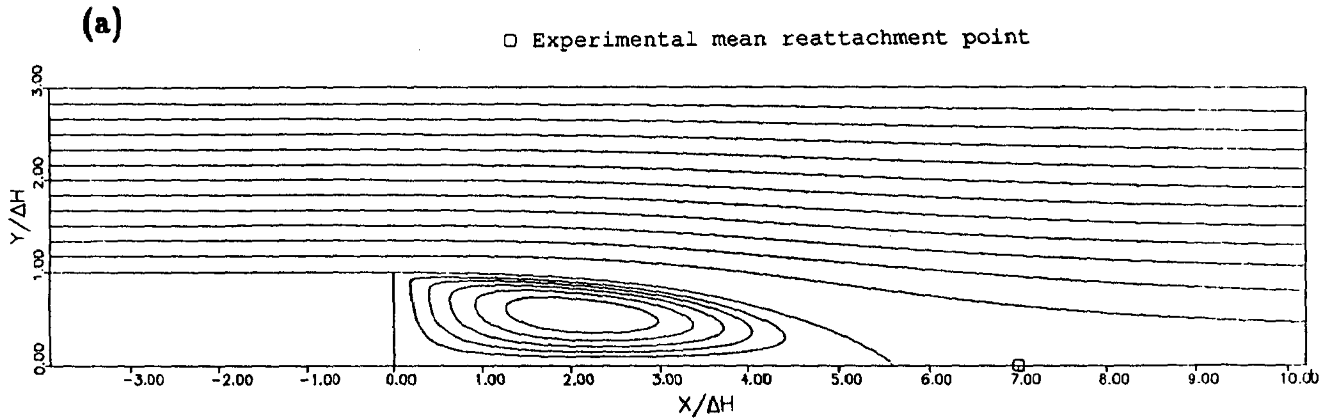

Figure 2a Comparison of the predictions of the standard K-ε model with the experiments of Kim et al (1980) for turbulent flow past a backward-facing step: mean-flow streamlines.

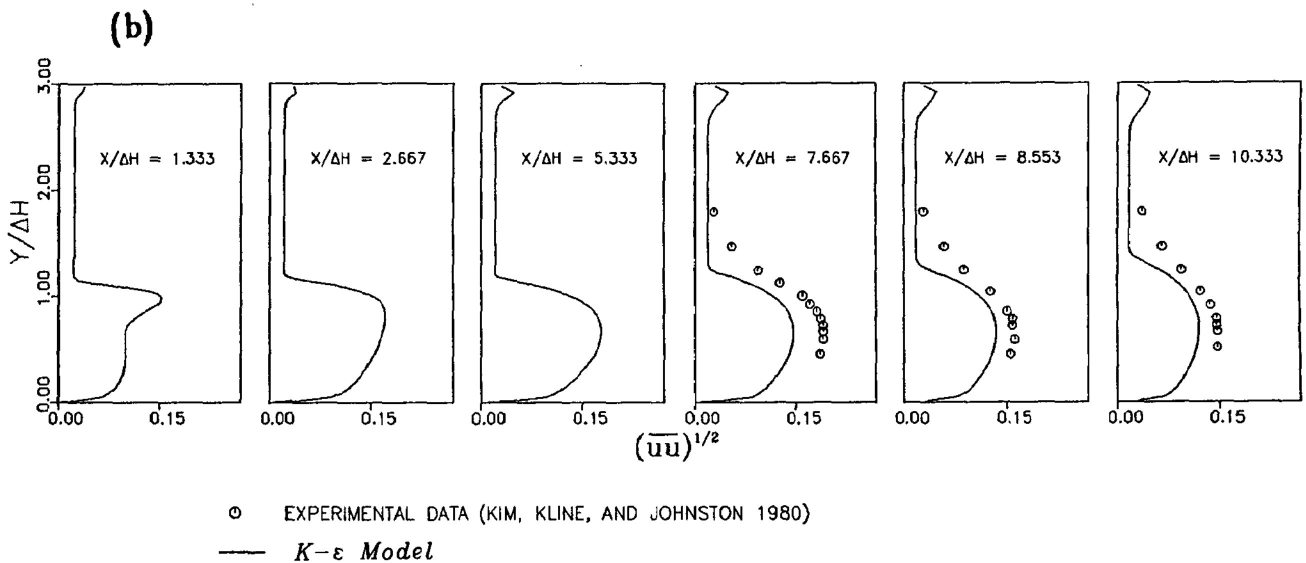

Figure 2b Comparison of the predictions of the standard K-ε model with the experiments of Kim et al (1980) for turbulent flow past a backward-facing step: turbulence-intensity profiles.

atically develop such models (with $\alpha_4 = 0$); he built up the length and time scales from the turbulent kinetic energy, turbulent dissipation rate, and the invariants of $\bar{S}_{ij}$ and $\bar{\omega}_{ij}$. Saffman (1977) proposed similar anisotropic models that were solved in conjunction with modeled transport equations for K and ω^2 (where $\omega \equiv \varepsilon/K$). Pope (1975) and Rodi (1976) developed alternative anisotropic eddy-viscosity models from the Reynolds-stress transport equation by making an equilibrium hypothesis. Yoshizawa (1984, 1987) derived a more complete two-equation model—with a nonlinear correction to the eddy viscosity of the full form of (66)—by means of a two-scale Direct Interaction Approximation (DIA) method. In this approach, Kraichnan's DIA formalism (cf Kraichnan 1964) is combined with a scale-expansion technique whereby the slow variations of the mean fields are distinguished from the fast variations of the fluctuating fields by means of a scale parameter. The length and time scales of the turbulence are built up from the turbulent kinetic energy and dissipation rate for which modeled transport equations are derived. These modeled transport equations are identical in form to (62c) and (62d) except for the addition of higher order cross-diffusion terms. The numerical values of the constants are derived directly from the theory (as with the RNG K-ε model). However, in applications it has been found that these values need to be adjusted (see Nisizima & Yoshizawa 1987).

Speziale (1987b) developed a nonlinear K-ε model based on a simplified version of (66) obtained by invoking the constraint of MFI in the limit of two-dimensional turbulence. In this model—where the length and time scales are built up from the turbulent kinetic energy and dissipation rate—the Reynolds stress tensor is taken to be of the form[3]

$$\tau_{ij} = \frac{2}{3}K\delta_{ij} - 2C_\mu \frac{K^2}{\varepsilon}\bar{S}_{ij} - 4C_D C_\mu^2 \frac{K^3}{\varepsilon^2}\left(\bar{S}_{ik}\bar{S}_{kj} - \frac{1}{3}\bar{S}_{mn}\bar{S}_{mn}\delta_{ij}\right) - 4C_E C_\mu^2 \frac{K^3}{\varepsilon^2}\left(\mathring{\bar{S}}_{ij} - \frac{1}{3}\mathring{\bar{S}}_{mm}\delta_{ij}\right), \quad (68)$$

where

$$\mathring{\bar{S}}_{ij} = \frac{\partial \bar{S}_{ij}}{\partial t} + \bar{u}_k \frac{\partial \bar{S}_{ij}}{\partial x_k} - \frac{\partial \bar{u}_i}{\partial x_k}\bar{S}_{kj} - \frac{\partial \bar{u}_j}{\partial x_k}\bar{S}_{ki} \quad (69)$$

is the frame-indifferent Oldroyd derivative of $\bar{S}_{ij}$, and $C_D = C_E = 1.68$.

[3] It is interesting to note that Rubinstein & Barton (1990) recently derived an alternative version of this model—which neglects the convective derivative in (69)—by using the RNG method of Yakhot & Orszag (1986).

Equation (68) can also be thought of as an approximation for turbulent flows where $\tau_0/T_0 \ll 1$, since MFI [which (68) satisfies identically] becomes exact in the limit as $\tau_0/T_0 \to 0$. This model bears an interesting resemblance to the Rivlin-Ericksen fluids of viscoelasticity; it has long been known that there are analogies between the mean turbulent flow of a Newtonian fluid and the laminar flow of a non-Newtonian fluid (cf Rivlin 1957). Speziale (1987b) and Speziale & Mac Giolla Mhuiris (1989a) showed that this model yields much more accurate predictions for the normal Reynolds-stress anisotropies in turbulent channel flow and homogeneous shear flow. [The standard K-ε model erroneously predicts that $\tau_{xx} = \tau_{yy} = \tau_{zz} = \frac{2}{3}K$.) As a result of this feature, the nonlinear K-ε model is capable of predicting turbulent secondary flows in noncircular ducts, unlike the standard K-ε model, which erroneously predicts a unidirectional mean turbulent flow (see Figures 3*a*–*c*). Comparably good predictions of turbulent secondary

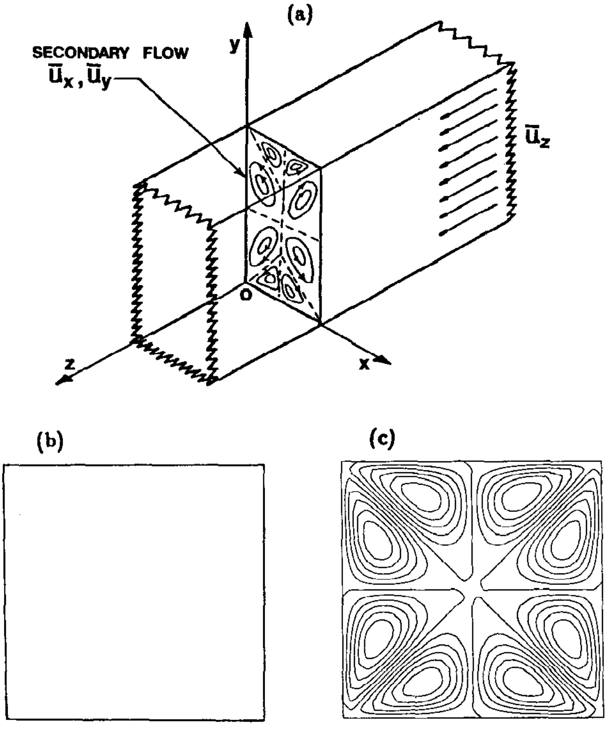

Figure 3 Turbulent secondary flow in a rectangular duct: (*a*) experiments; (*b*) standard K-ε model; (*c*) nonlinear K-ε model of Speziale (1987b).

flows in a rectangular duct were obtained much earlier by Launder & Ying (1972), Gessner & Po (1976), and Demuren & Rodi (1984), who used the nonlinear algebraic Reynolds-stress model of Rodi (1976). As a result of the more accurate prediction of normal Reynolds-stress anisotropies—and the incorporation of weak relaxation effects—the nonlinear K-ε model of Speziale (1987b) was also able to yield improved results for turbulent flow past a backward-facing step (compare Figures 4*a,b* with Figures 2*a,b*). Most notably, the nonlinear K-ε model predicts reattachment at $x/\Delta H \doteq 6.4$—a value that is more in line with the experimental value of $x/\Delta H \doteq 7.0$. (As shown earlier, the standard K-ε model yields a value of $x/\Delta H \doteq 5.7$ when a single-layer log wall region is used.) While these improvements are encouraging, it must be mentioned that the nonlinear K-ε model still has many of the same deficiencies of the simpler two-equation models—namely, the inability to properly account for component Reynolds-stress relaxation effects or body-force effects.

Alternative two-equation models based on the solution of a modeled transport equation for an integral length scale (the K-l model) or the reciprocal time scale (the K-ω model) have also been considered during the past 15 years. In the K-l model (see Mellor & Herring 1973) a modeled transport equation is solved for the integral length scale l, defined by

$$l(\mathbf{x}, t) = \frac{1}{2K}\int_{-\infty}^{\infty} \frac{R_{ii}(\mathbf{x}, \mathbf{r}, t)}{4\pi|\mathbf{r}|^2}\, d^3r, \tag{70}$$

where $R_{ij} \equiv \overline{u_i'(\mathbf{x}, t)u_j'(\mathbf{x}+\mathbf{r}, t)}$ is the two-point velocity-correlation tensor. The typical form of the modeled transport equation for l is as follows:

$$\frac{\partial(Kl)}{\partial t} + \bar{u}_i\frac{\partial(Kl)}{\partial x_i} = \frac{\partial}{\partial x_i}\left[(\nu + \beta_1 K^{1/2} l)\frac{\partial}{\partial x_i}(Kl) + \beta_2 K^{3/2} l \frac{\partial l}{\partial x_i}\right] - \beta_3 l \tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - \beta_4 K^{3/2}, \tag{71}$$

where β_1, ..., β_4 are empirical constants. Equation (71) is derived by integrating the contracted form of a modeled transport equation for the two-point velocity-correlation tensor R_{ij} (see Wolfshtein 1970). Mellor and coworkers have utilized this K-l model—with an eddy viscosity of the form (52)—in the solution of a variety of engineering and geophysical fluid-dynamics problems [see Mellor & Herring (1973) and Mellor & Yamada (1974) for a more thorough review]. It has been argued—and correctly so—that it is sounder to base the turbulent macroscale on the integral length scale (70) rather than on the turbulent dissipation rate, which only formally determines the turbulent microscale. However, for homogeneous turbulent flows, it is a simple matter to show that this *K-l* model is *equivalent* to a *K*-ε model where the constants C_μ, $C_{\varepsilon 1}$, and $C_{\varepsilon 2}$ assume slightly different values (cf Speziale 1990). Furthermore, the modeled transport equation (71) for *l* has comparable problems to the modeled ε-transport equation in so far as integrations to a solid boundary are concerned. (Either wall functions or wall damping must be used.) Consequently, at the current stage of development, it does not appear that this type of *K-l* model offers any significant advantages over the *K*-ε model.

Wilcox and coworkers have developed two-equation models of the *K*-ω type (see Wilcox & Traci 1976, Wilcox 1988). In this approach, modeled transport equations are solved for the turbulent kinetic energy *K* and reciprocal turbulent time scale $\omega \equiv \varepsilon/K$. The modeled transport equation for ω is of the form

$$\frac{\partial \omega}{\partial t}+\bar{u}_i\frac{\partial \omega}{\partial x_i} = -\gamma_1\frac{\omega}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j}-\gamma_2\omega^2+\frac{\partial}{\partial x_i}\left(\frac{\nu_T}{\sigma_\omega}\frac{\partial \omega}{\partial x_i}\right)+\nu\nabla^2\omega, \tag{72}$$

where $\nu_T = \gamma^* K/\omega$, and γ_1, γ_2, γ^*, and σ_ω are constants. Equation (72) is obtained by making the same type of assumptions in the modeling of the exact transport equation for ω that were made in developing the modeled ε-transport equation (62d). For homogeneous turbulent flows, there is little difference between the *K*-ε and *K*-ω models. The primary difference between the two models is in their treatment of the transport terms: The *K*-ε model is based on a gradient transport hypothesis for ε, whereas the *K*-ω model uses the same hypothesis for ω instead. Despite the fact that ω is singular at a solid boundary, there is some evidence to suggest that the *K*-ω model is more computationally robust for the integration of turbulence models to a wall (i.e. there is the need for less empirical damping; see Wilcox 1988).

SECOND-ORDER CLOSURE MODELS

Theoretical Background

Although two-equation models are the first simple and complete Reynolds-stress models to be developed, they still have significant deficiencies that make their application to complex turbulent flows precarious. As mentioned earlier, the two-equation models of the eddy-viscosity type have the following major deficiencies: (*a*) the inability to properly account for streamline curvature, rotational strains, and other body-force effects; and (*b*) the neglect of nonlocal and history effects on the Reynolds-stress anisotropies. Most of these deficiencies are intimately tied to the assumption that there is a clear-cut separation of scales at the second-moment

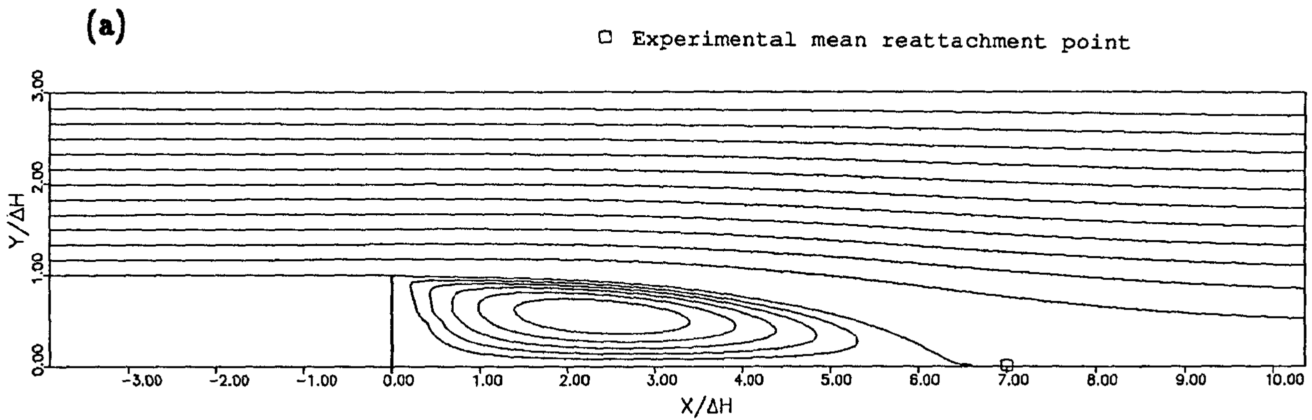

Figure 4a Comparison of the predictions of the nonlinear K-ε model of Speziale (1987b) with the experiments of Kim et al (1980) for turbulent flow past a backward-facing step: mean-flow streamlines.

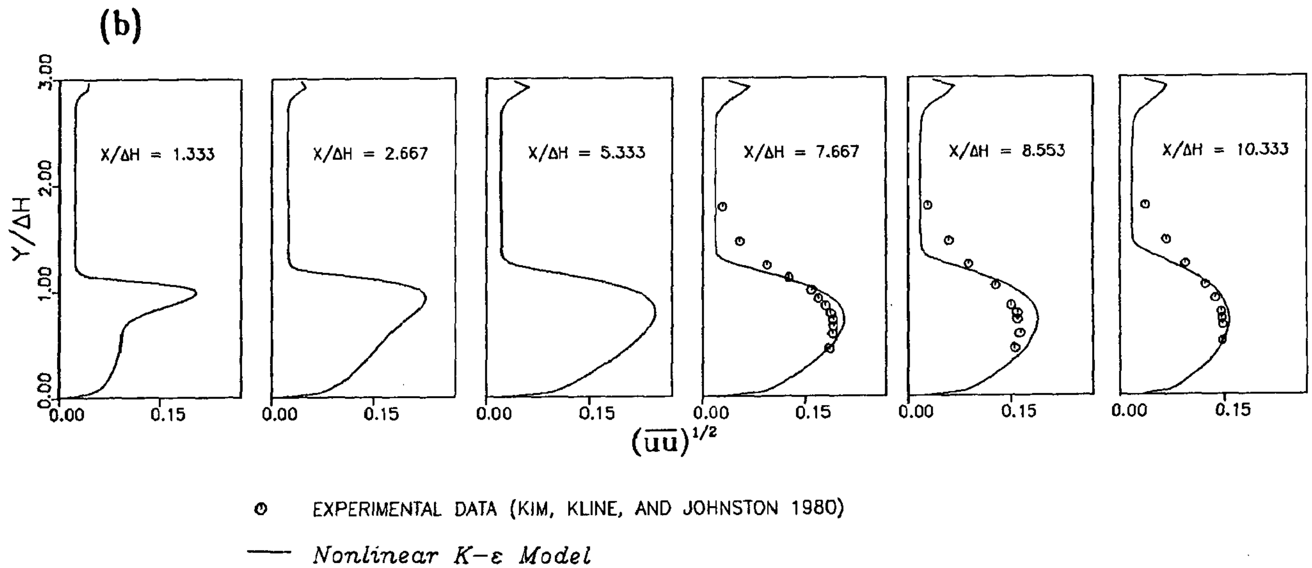

Figure 4b Comparison of the predictions of the nonlinear K-ε model of Speziale (1987b) with the experiments of Kim et al (1980) for turbulent flow past a backward-facing step: turbulence-intensity profiles.

level (i.e. the level of the Reynolds-stress tensor). This can best be illustrated by the example of homogeneous shear flow presented in the previous section. For this problem, the equilibrium value of the ratio of fluctuating to mean time scales is given by

$$\frac{\tau_0}{T_0} \equiv \frac{SK}{\varepsilon} \doteq 4.8$$

for the K-ε model. This is in flagrant conflict with the assumption that $\tau_0/T_0 \ll 1$, which was crucial for the derivation of the K-ε model! While some of the deficiencies cited above can be partially overcome by the use of two-equation models with a nonlinear algebraic correction to the eddy viscosity, major improvements can only be achieved by higher order closures—the simplest of which are second-order closure models.

Second-order closure models are based on the Reynolds-stress transport equation (23). Since this equation automatically accounts for the convection and diffusion of Reynolds stresses, second-order closure models (unlike eddy-viscosity models) are able to account for strong nonlocal and history effects. Furthermore, since the Reynolds-stress transport equation contains convection and production terms that adjust themselves automatically in turbulent flows with streamline curvature or a system rotation (through the addition of scale factors or Coriolis terms), complex turbulent flows involving these effects are usually better described.

In order to develop a second-order closure, models must be provided for the higher order correlations C_{ijk}, Π_{ij}, and ε_{ij} on the right-hand side of the Reynolds-stress transport equation (23). These terms, sufficiently far from solid boundaries, are typically modeled as follows:

1. The third-order transport term C_{ijk} is modeled by a gradient transport hypothesis that is based on the usual assumption that there is a clear-cut separation of scales between mean and fluctuating fields.
2. The pressure-strain correlation Π_{ij} and the dissipation-rate correlation ε_{ij} are modeled based on ideas from homogeneous turbulence, wherein the departures from isotropy are assumed to be small enough to allow for a Taylor-series expansion about a state of isotropic turbulence.

Near solid boundaries, either wall functions or wall damping are used in a comparable manner to that discussed in the last section. One important point to note is that the crucial assumption of separation of scales is made only at the *third-moment level*. This leads us to the raison d'être of second-order closure modeling: *Since crude approximations for the second moments in eddy-viscosity models often yield adequate approximations for first-order moments* (i.e. $\bar{\mathbf{u}}$ *and* $\bar{\mathrm{p}}$), *it may follow that crude approximations for the* *third-order moments can yield adequate approximations for the second-order moments in Reynolds-stress transport models.*

The pressure-strain correlation Π_{ij} plays a crucial role in determining the structure of most turbulent flows. Virtually all of the models for Π_{ij} that have been used in conjunction with second-order closure models are based on the assumption of local homogeneity. For homogeneous turbulent flows, the pressure-strain correlation takes the form

$$\Pi_{ij} = A_{ij} + M_{ijkl}\frac{\partial \bar{u}_k}{\partial x_l}, \tag{73}$$

where

$$A_{ij} = \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\frac{\partial^2 u_k' u_l'}{\partial y_k \partial y_l}\frac{d^3 y}{|\mathbf{x}-\mathbf{y}|}},$$

$$M_{ijkl} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right)\frac{\partial u_l'}{\partial y_k}\frac{d^3 y}{|\mathbf{x}-\mathbf{y}|}}.$$

Here, the first term on the right-hand side of (73) is referred to as the slow pressure-strain, whereas the second term is called the rapid pressure-strain. It has been shown that A_{ij} and M_{ijkl} are functionals—in time and wavenumber space—of the energy-spectrum tensor (cf Weinstock 1981, Reynolds 1987). In a one-point closure, this suggests models for A_{ij} and M_{ijkl} that are functionals of the Reynolds-stress tensor and turbulent dissipation rate. Neglecting history effects, the simplest such models are of the form

$$A_{ij} = \varepsilon \mathscr{A}_{ij}(\mathbf{b}), \tag{74}$$

$$M_{ijkl} = K\mathscr{M}_{ijkl}(\mathbf{b}). \tag{75}$$

These algebraic models—based on the assumptions stated above—are obtained by using simple dimensional arguments combined with the fact that Π_{ij} vanishes in the limit of isotropic turbulence [a constraint identically satisfied if $\mathscr{A}_{ij}(0) = 0$ and $\mathscr{M}_{ijkl}(0) = 0$]. Essentially every model for the pressure-strain correlation that has been used in second-order closures is of the form (73)–(75).

Lumley (1978) was probably the first to systematically develop a general representation for the pressure-strain correlation based on (73)–(75). It can be shown that invariance under a change of coordinates—coupled with the assumption of analyticity about the isotropic state $b_{ij} = 0$—restricts (73) to be of the form[4]

$$\Pi_{ij} = a_0\varepsilon b_{ij} + a_1\varepsilon(b_{ik}b_{kj} - \tfrac{1}{3}\mathrm{II}\delta_{ij}) + a_2 K\bar{S}_{ij} + (a_3 \mathrm{tr}\,\mathbf{b}\cdot\bar{\mathbf{S}} + a_4 \mathrm{tr}\,\mathbf{b}^2\cdot\bar{\mathbf{S}})Kb_{ij} + (a_5 \mathrm{tr}\,\mathbf{b}\cdot\bar{\mathbf{S}} + a_6 \mathrm{tr}\,\mathbf{b}^2\cdot\bar{\mathbf{S}})K(b_{ik}b_{kj} - \tfrac{1}{3}\mathrm{II}\delta_{ij}) + a_7 K(b_{ik}\bar{S}_{jk} + b_{jk}\bar{S}_{ik} - \tfrac{2}{3}\mathrm{tr}\,\mathbf{b}\cdot\bar{\mathbf{S}}\delta_{ij}) + a_8 K(b_{ik}b_{kl}\bar{S}_{jl} + b_{jk}b_{kl}\bar{S}_{il} - \tfrac{2}{3}\mathrm{tr}\,\mathbf{b}^2\cdot\bar{\mathbf{S}}\delta_{ij}) + a_9 K(b_{ik}\bar{\omega}_{jk} + b_{jk}\bar{\omega}_{ik}) + a_{10}K(b_{ik}b_{kl}\bar{\omega}_{jl} + b_{jk}b_{kl}\bar{\omega}_{il}), \quad (76)$$

where

$$a_i = a_i(\mathrm{II}, \mathrm{III}), \qquad i = 0, 1, \ldots, 10,$$

$$\mathrm{II} = b_{ij}b_{ij}, \qquad \mathrm{III} = b_{ik}b_{kl}b_{li},$$

and $\mathrm{tr}(\cdot)$ denotes the trace. The eigenvalues $b^{(\alpha)}$ of b_{ij} are bounded as follows (see Lumley 1978)

$$-\tfrac{1}{3} \le b^{(\alpha)} \le \tfrac{2}{3}, \qquad \alpha = 1, 2, 3, \quad (77)$$

and for many engineering flows, $\|\mathbf{b}\|_2 \equiv |b^{(\alpha)}|_{\max} < 0.25$. Hence, it would seem that a low-order Taylor-series truncation of (76) could possibly provide an adequate approximation. To the first order in b_{ij}, one has

$$\Pi_{ij} = -C_1\varepsilon b_{ij} + C_2 K\bar{S}_{ij} + C_3 K(b_{ik}\bar{S}_{jk} + b_{jk}\bar{S}_{ik} - \tfrac{2}{3}b_{mn}\bar{S}_{mn}\delta_{ij}) + C_4 K(b_{ik}\bar{\omega}_{jk} + b_{jk}\bar{\omega}_{ik}), \quad (78)$$

which is the form used in the Launder, Reece & Rodi (1975) model. In this model (henceforth labeled LRR), the constants C_1, C_3, and C_4 were calibrated based on the results of return to isotropy and homogeneous shear-flow experiments. The constant C_2 was chosen to be consistent with the value obtained by Crow (1968) from RDT for an irrotationally strained turbulence starting from an initially isotropic state. This yielded the following values for the constants in the simplified version of the LRR model: $C_1 = 3.6$, $C_2 = 0.8$, $C_3 = 0.6$, and $C_4 = 0.6$. It should be noted that the representation for the slow pressure-strain correlation in the LRR model is the Rotta (1951) return-to-isotropy model with the Rotta constant C_1 adjusted from 2.8 to 3.6 (a value that is in the range of what can be extrapolated from physical experiments). This model—consistent with

[4] This representation, obtained by using the results of Smith (1971) on isotropic tensor functions, is actually somewhat more compact than that obtained by Lumley and coworkers.

experiments—predicts that an initially anisotropic, homogeneous turbulence relaxes gradually to an isotropic state after the mean-velocity gradients are removed.

An even simpler version of (78) was proposed by Rotta (1972) wherein $C_3 = C_4 = 0$. This model has been used by Mellor and coworkers for the calculation of many engineering and geophysical flows (see Mellor & Herring 1973, Mellor & Yamada 1974). Research during the past decade has focused attention on the development of nonlinear models for Π_{ij}. Lumley (1978) and Shih & Lumley (1985) developed a nonlinear model by using the constraint of realizability discussed earlier. Haworth & Pope (1986) developed a nonlinear model for the pressure-strain correlation based on the Langevin equation used in the pdf description of turbulence. This model—which was cubic in the anisotropy tensor—was calibrated based on homogeneous-turbulence experiments and was shown to perform quite well for a range of such flows. Speziale (1987a) developed a hierarchy of second-order closure models that were consistent with the MFI constraint in the limit of two-dimensional turbulence.[5] [This constraint was also used by Haworth & Pope (1986) in the development of their second-order closure.] Launder and coworkers (cf Fu et al 1987, Craft et al 1989) have developed new nonlinear models for the pressure-strain correlation based on the use of realizability combined with a calibration using newer homogeneous-turbulence experiments. W. C. Reynolds (private communication, 1988) has attempted to develop models that are consistent with RDT, and the present author has been analyzing models based on a dynamical-systems approach (see Speziale & Mac Giolla Mhuiris 1989a,b, Speziale et al 1990).

The modeling of the dissipation-rate tensor, at high turbulence Reynolds numbers, is usually based on the Kolmogorov hypothesis of isotropy in which

$$\varepsilon_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij}, \tag{79}$$

given that $\varepsilon \equiv \nu\overline{\partial u_i'/\partial x_j \partial u_i'/\partial x_j}$ is the scalar dissipation rate. Here, the turbulent dissipation rate ε is typically taken to be a solution of the modeled transport equation

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K} + C_{\varepsilon}\frac{\partial}{\partial x_i}\left(\frac{K}{\varepsilon}\tau_{ij}\frac{\partial \varepsilon}{\partial x_j}\right) + \nu\nabla^2\varepsilon, \tag{80}$$

where $C_{\varepsilon 1} = 1.44$, $C_{\varepsilon 2} = 1.92$, and $C_\varepsilon = 0.15$. Equation (80) is identical to

[5] MFI in the limit of two-dimensional turbulence can be satisfied identically by (76) if $a_{10} = -3a_9 + 12$; see Speziale (1987a).

the ε-transport equation used in the K-ε model, with one exception: The turbulent diffusion term is *anisotropic*. Hence, the logic used in deriving (80) is virtually the same as that used in deriving the modeled ε-transport equation for the K-ε model. Near solid boundaries, anisotropic corrections to (79) have been proposed that are typically of the algebraic form (see Hanjalić & Launder 1976)

$$\varepsilon_{ij} = \tfrac{2}{3}\varepsilon\delta_{ij} + 2\varepsilon f_s b_{ij}, \tag{81}$$

where f_s is a function of the turbulence Reynolds number $\mathrm{Re}_t \equiv K^2/\nu\varepsilon$. Equation (81)—which can be thought of as a first-order Taylor-series expansion about a state of isotropic turbulence—is solved in conjunction with (80), where the model coefficients are taken to be functions of Re_t as a solid boundary is approached (cf Hanjalić & Launder 1976). Since the commonly used models for the deviatoric part of ε_{ij} are similar in form to the first term in (76), it is possible to use the isotropic model (79) and then model the deviatoric part of ε_{ij} together with the pressure-strain correlation, as was first pointed out by Lumley (1978). Also, as an alternative to (81), the isotropic form (79) can be used in a wall-bounded flow if suitable wall functions are used to bridge the outer and inner flows.

One major weakness of the models (80)–(81) is their neglect of rotational strains. For example, in a rotating isotropic turbulence, the modeled ε-transport equation (80) yields the *same decay rate* independent of the rotation rate of the reference frame. In stark contrast to this result, physical and numerical experiments indicate that the decay rate of the turbulent kinetic energy can be considerably reduced by a system rotation—the inertial waves generated by the rotation disturb the phase coherence needed to cascade energy from the large scales to the small scales (see Wigeland & Nagib 1978, Speziale et al 1987). A variety of modifications to (80) have been proposed during the last decade to account for rotational strains (see Pope 1978, Hanjalić & Launder 1980, Bardina et al 1985). However, these modifications tended to be "one-problem" corrections that gave rise to difficulties when other flows were considered. It was recently shown by the present author that all of these modified ε-transport equations are more ill behaved than the standard model (80) for general homogeneous turbulent flows in a rotating frame (e.g. they fail to properly account for the stabilizing effect of a strong system rotation on a homogeneously strained turbulent flow; see Speziale 1990).

At this point it should be mentioned that in the second-order closure models of Mellor and coworkers, the dissipation rate is modeled as in Equation (55), and a modeled transport equation for the integral length scale (70) is solved that is identical in form to (71). When this model has been applied to wall-bounded turbulent flows it has typically been used in conjunction with wall functions. In addition, it should also be mentioned that second-order closure models along the lines of the K-ω model of Wilcox and coworkers have been considered. [Here a modeled transport equation for the reciprocal time scale $\omega \equiv \varepsilon/K$ is solved (cf Wilcox 1988).]

In order to complete these second-order closures, a model for the third-order diffusion correlation C_{ijk} is needed. Since this is a third-order moment, the simplifying assumption of gradient transport (which is generally valid only when there is a clear-cut separation of scales) is typically made. Hence, all of the commonly used second-order closures are based on models for C_{ijk} of the form

$$C_{ijk} = -\mathscr{C}_{ijklmn}\frac{\partial \tau_{lm}}{\partial x_n},$$

where the diffusion tensor $\mathscr{C}_{ijklmn}$ can depend anisotropically on τ_{ij}. For many incompressible turbulent flows, the pressure-diffusion terms in C_{ijk} can be neglected in comparison to the triple velocity correlation $\overline{u_i u_j u_k}$. Then, the symmetry of C_{ijk} under an interchange of any of its three indices immediately yields the form

$$C_{ijk} = -C_s\frac{K}{\varepsilon}\left(\tau_{im}\frac{\partial \tau_{jk}}{\partial x_m}+\tau_{jm}\frac{\partial \tau_{ik}}{\partial x_m}+\tau_{km}\frac{\partial \tau_{ij}}{\partial x_m}\right), \tag{82}$$

which was first obtained by Launder, Reece & Rodi (1975) via an alternative analysis based on the transport equation for $\overline{u_i u_j u_k}$. Equation (82) is sometimes used in its isotropized form

$$C_{ijk} = -\frac{2}{3}C_s\frac{K^2}{\varepsilon}\left(\frac{\partial \tau_{jk}}{\partial x_i}+\frac{\partial \tau_{ik}}{\partial x_j}+\frac{\partial \tau_{ij}}{\partial x_k}\right) \tag{83}$$

(cf Mellor & Herring 1973). The constant C_s was chosen to be 0.11 by Launder, Reece & Rodi (1975) based on comparisons with experiments on thin shear flows. Similar models for C_{ijk} have been derived by Lumley (1978) from first principles (see also Lumley & Khajeh-Nouri 1974).

Examples

Now, by the use of some illustrative examples, a case is made for the superior predictive capabilities of second-order closures in comparison to zero-, one-, and two-equation models. First, to demonstrate the ability of second-order closure models to describe Reynolds-stress relaxation effects, we consider the return-to-isotropy problem. In this problem, an initially anisotropic, homogeneous turbulence—generated by the application of constant mean-velocity gradients—gradually relaxes to a state of isotropy after the mean-velocity gradients are removed. By introducing the transformed dimensionless time τ (where $d\tau = \varepsilon\, dt/2K$), the modeled Reynolds-stress transport equation can be written in the equivalent form

$$\frac{db_{ij}}{d\tau} = 2b_{ij} + \mathscr{A}_{ij}, \tag{84}$$

where $\mathscr{A}_{ij}$ is the dimensionless slow pressure-strain correlation. Since the rapid pressure-strain and transport terms vanish in this problem—and since the dissipation rate can be absorbed into the dimensionless time τ—only a model for the slow pressure-strain correlation is needed, as indicated in (84). In Figure 5, the predictions of the LRR model (where $\mathscr{A}_{ij} = -C_1 b_{ij}$ and the Rotta constant $C_1 = 3.0$) for the time evolution of the anisotropy tensor are compared with the experimental data of Choi & Lumley (1984) for the relaxation from plane-strain case. It is clear from this figure that this simple second-order closure model does a reasonably good job in reproducing the experimental trends, which show a gradual return to isotropy (where $b_{ij} \to 0$ as $\tau \to \infty$). This is in considerable contrast to eddy-viscosity (or nonlinear algebraic stress) models, which erroneously predict

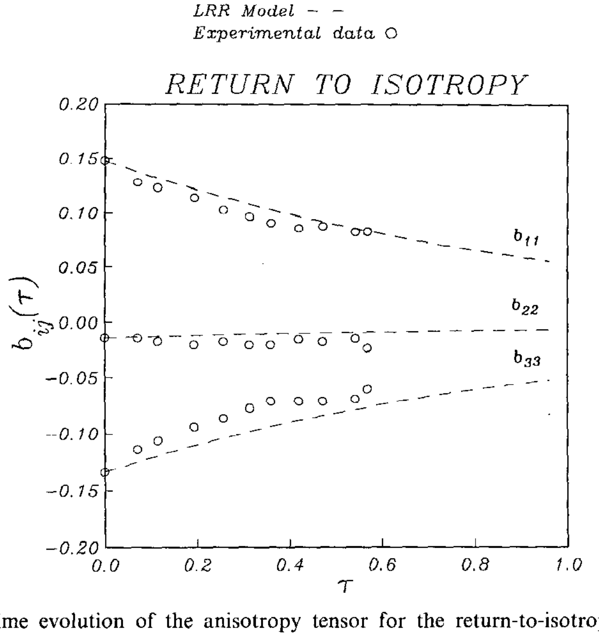

Figure 5 Time evolution of the anisotropy tensor for the return-to-isotropy problem: comparison of the predictions of the Launder, Reece & Rodi (LRR) model with the experimental data of Choi & Lumley (1984) for the relaxation from plane strain.

that $b_{ij} = 0$ for $\tau > 0$! Further improvements can be obtained with second-order closures based on nonlinear models for the slow pressure-strain correlation. A simple quadratic model was recently proposed by Sarkar & Speziale (1990) in which

$$\mathscr{A}_{ij} = -C_1 b_{ij} + C_2(b_{ik}b_{kj} - \tfrac{1}{3}\mathrm{II}\delta_{ij}), \tag{85}$$

with $C_1 = 3.4$ and $C_2 = 4.2$. This model does a better job in reproducing the trends of the Choi & Lumley (1984) experiment (see Figure 6). Most notably, the quadratic model (85) yields curved trajectories in the ξ-η phase space (where $\xi = \mathrm{III}^{1/3}$, $\eta = \mathrm{II}^{1/2}$) that are well within the range of experimental data; any linear or quasi-linear model where $C_2 = 0$ erroneously yields straight-line trajectories in the ξ-η phase space, as is clearly shown in Figure 6.

As alluded to earlier, second-order closure models perform far better than eddy-viscosity models in rotating turbulent flows. To illustrate this

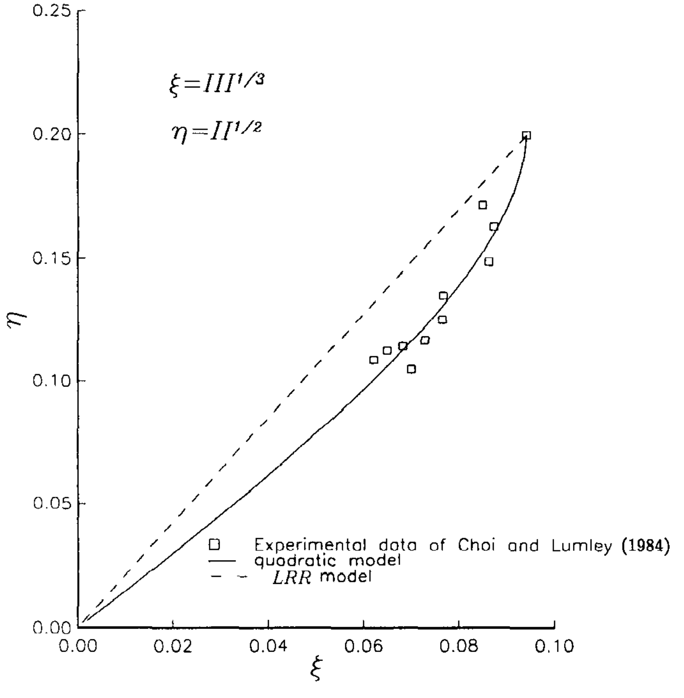

Figure 6 Phase-space portrait of the return-to-isotropy problem: comparison of the predictions of the LRR model and of the quadratic model of Sarkar & Speziale (1990) with the experimental data of Choi & Lumley (1984) for the relaxation from plane strain.

point, a comparison of the predictions of the standard K-ε model and the LRR model is now made for the problem of homogeneous turbulent shear flow in a rotating frame. This problem represents a nontrivial test of turbulence models, since a system rotation can have either a stabilizing or a destabilizing effect on turbulent shear flow. We focus here on the most basic type of plane shear flow in a rotating frame, in which

$$\frac{\partial \bar{u}_i}{\partial x_j} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_i = (0, 0, \Omega) \tag{86}$$

(see Figure 7). For the case of pure shear flow ($\Omega = 0$), the LRR model yields substantially improved predictions over the K-ε model for the equilibrium values of b_{ij} and SK/ε, as shown in Table 1. Since the standard K-ε model is frame indifferent, it erroneously yields solutions for rotating shear flow that are independent of Ω. Second-order closure models, on the other hand, yield rotationally dependent solutions owing to the effect of the Coriolis acceleration. For any homogeneous turbulent flow in a rotating frame, second-order closure models take the form (cf Speziale 1989)

$$\dot{\tau}_{ij} = -\tau_{ik}\frac{\partial \bar{u}_j}{\partial x_k} - \tau_{jk}\frac{\partial \bar{u}_i}{\partial x_k} + \Pi_{ij} - \varepsilon_{ij} - 2(\tau_{ik}e_{mkj}\Omega_m + \tau_{jk}e_{mki}\Omega_m), \tag{87}$$

where the mean-vorticity tensor $\bar{\omega}_{ij}$ in the model for Π_{ij} [see Equation (78)] is replaced with the intrinsic mean-vorticity tensor $\overline{W}_{ij}$ defined in (32). The equations of motion for the LRR model are obtained by substituting (86) into (87) and the modeled ε-transport equation

$$\dot{\varepsilon} = -C_{\varepsilon 1}\frac{\varepsilon}{K}\tau_{ij}\frac{\partial \bar{u}_i}{\partial x_j} - C_{\varepsilon 2}\frac{\varepsilon^2}{K}, \tag{88}$$

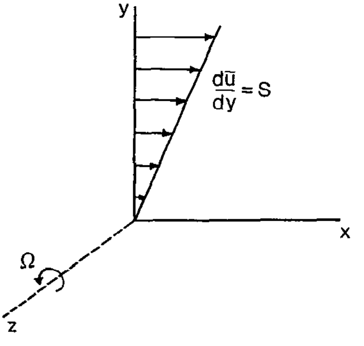

Figure 7 Homogeneous shear flow in a rotating frame.

Table 1 Comparison of the predictions of the standard K-ε model and the Launder, Reece & Rodi (LRR) model with the experiments of Tavoularis & Corrsin (1981) on homogeneous shear flow

Equilibrium values	Standard K-ε model	LRR model	Experiments
$(b_{11})_\infty$	0	0.193	0.201
$(b_{22})_\infty$	0	−0.096	−0.147
$(b_{12})_\infty$	−0.217	−0.185	−0.150
$(SK/\varepsilon)_\infty$	4.82	5.65	6.08

which is not directly affected by rotations. A complete dynamical-systems analysis of these nonlinear ordinary differential equations—which are typically solved for initial conditions that correspond to a state of isotropic turbulence—was conducted recently by Speziale & Mac Giolla Mhuiris (1989a). It was found that ε/SK and b_{ij} have bounded equilibrium values that are independent of the initial conditions and only depend on Ω and S through the dimensionless ratio Ω/S. There are two equilibrium solutions for $(\varepsilon/SK)_\infty$: one where $(\varepsilon/SK)_\infty = 0$, which exists for *all* Ω/S; and one where $(\varepsilon/SK)_\infty > 0$, which only exists for an intermediate band of Ω/S (see Figure 8*a*). The trivial equilibrium solution is predominantly associated with solutions for K and ε that undergo a power-law decay with time; the nonzero equilibrium solution (ellipse ACB on the bifurcation diagram shown in Figure 8*a*) is associated with unstable flow wherein K and ε undergo an exponential time growth at the same rate. The two solutions exchange stabilities in the interval AB (i.e. this is a degenerate type of transcritical bifurcation). In stark contrast to these results, the standard K-ε model erroneously predicts the *same* equilibrium value for $(\varepsilon/SK)_\infty$ independent of Ω/S (see Figure 8*b*). In Figures 9*a–c*, the time evolution of the turbulent kinetic energy predicted by the standard K-ε model and the LRR model are compared with the large-eddy simulations of Bardina et al (1983). It is clear that the second-order closure model is able to properly account for the stabilizing or destabilizing effect of rotations on shear flow, whereas the K-ε model (as well as the nonlinear K-ε model, whose predictions are almost identical for rotating shear flow) erroneously predicts results that are independent of the rotation rate Ω. The LRR model predicts that there is unstable flow (where K and ε grow exponentially) only for rotation rates lying in the intermediate range $-0.1 \leq \Omega/S \leq 0.39$, whereas linear-stability analyses indicate unstable flow for $0 \leq \Omega/S \leq 0.5$.

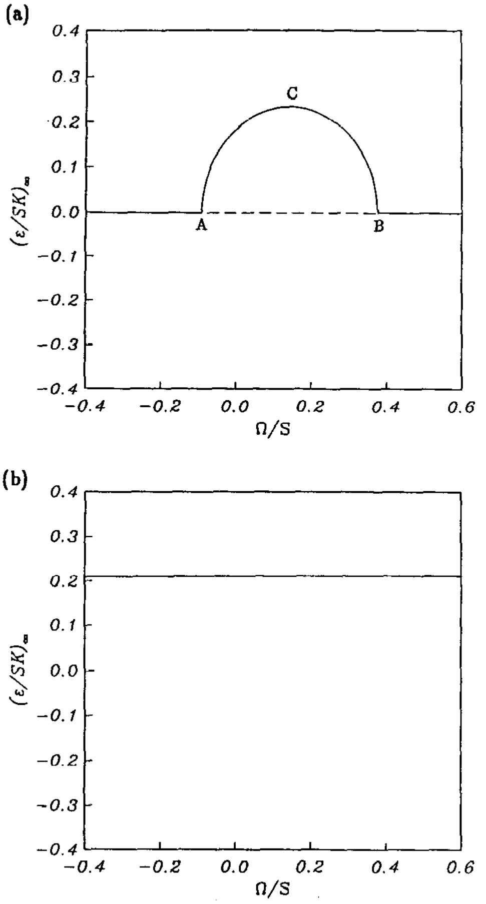

Figure 8 Bifurcation diagram for rotating shear flow: (*a*) LRR model; (*b*) standard K-ε model.

Similar improved results using second-order closures have been recently obtained by Gatski & Savill (1989) for curved homogeneous shear flow.

Finally, an example of an inhomogeneous wall-bounded turbulent flow is given. The problem of rotating channel flow recently considered by Launder et al (1987) represents a challenging example. In this problem a turbulent channel flow is subjected to a steady spanwise rotation (see Figure 10). Physical and numerical experiments (see Johnston et al 1972, Kim 1983) indicate that Coriolis forces arising from a system rotation cause the mean-velocity profile $\bar{u}(y)$ to become asymmetric about the

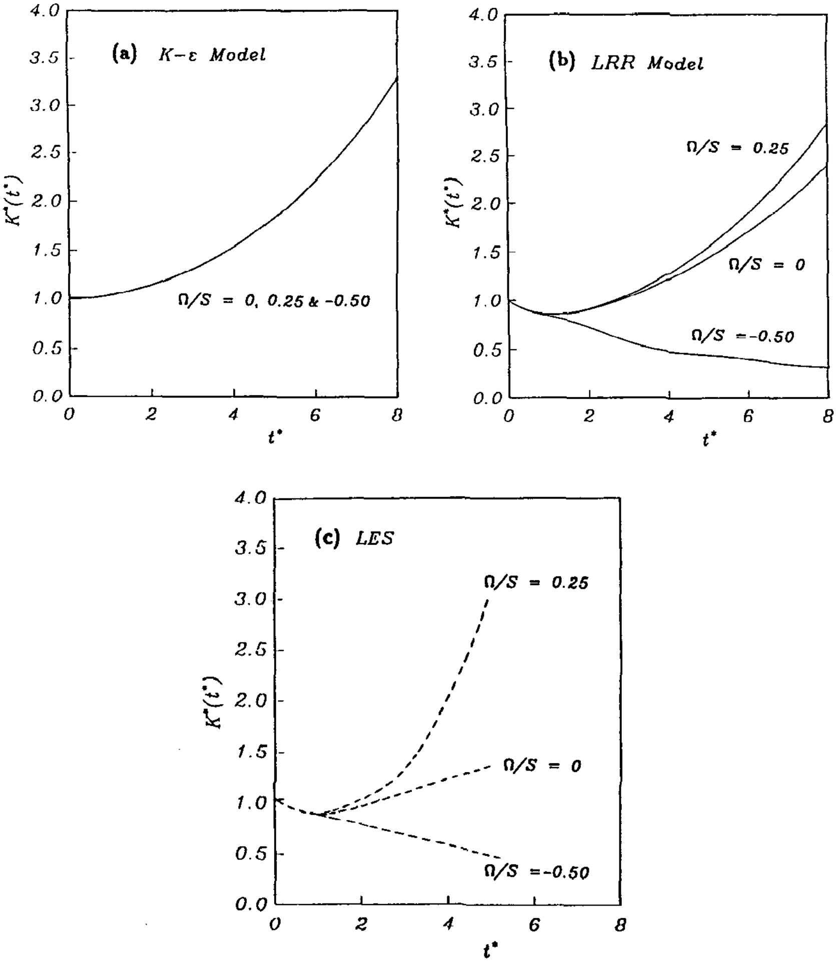

Figure 9 Time evolution of the turbulent kinetic energy in rotating shear flow for $\varepsilon_0/SK_0 = 0.296$: (*a*) standard *K*-ε model; (*b*) LRR model; and (*c*) large-eddy simulations (LES) of Bardina et al (1983).

channel centerline. In Figure 11, the mean-velocity profile computed by Launder et al (1987) using the Gibson & Launder (1978) second-order closure model is compared with the results of the *K*-ε model and the experimental data of Johnston et al (1972) for a Reynolds number Re = 11,500 and a rotation number Ro = 0.21. From this figure, it is clear that the second-order closure model yields a highly asymmetric mean-

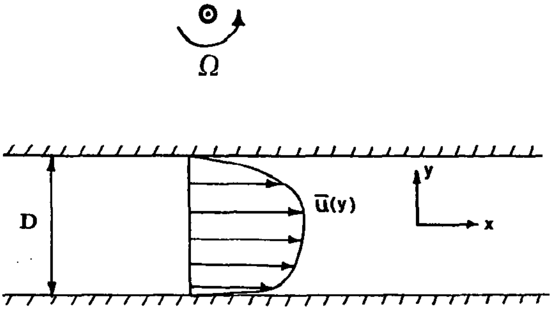

Figure 10 Fully developed turbulent channel flow in a rotating frame.

velocity profile that is well within the range of the experimental data. The standard K-ε model erroneously predicts the *same* symmetric mean-velocity profile as in an inertial frame (where $\Omega = 0$), as shown in Figure 11. Comparable improvements in the prediction of curved turbulent shear flows have been obtained by Gibson & Rodi (1981) and Gibson & Younis (1986) using second-order closure models. Likewise, turbulent flows

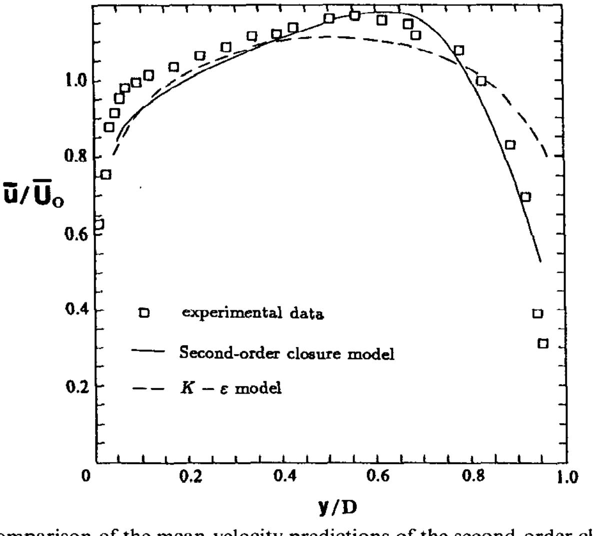

Figure 11 Comparison of the mean-velocity predictions of the second-order closure model of Gibson & Launder (1978) and the standard K-ε model with the experimental data of Johnston et al (1972) on rotating channel flow (partially taken from Launder et al 1987).

involving buoyancy effects have been shown to be better described by second-order closure models (cf Zeman & Lumley 1976, 1979). In these problems, the Coriolis terms on the right-hand side of the Reynolds-stress transport equation (87) are replaced with the body-force term

$$\beta(g_i\overline{T'u'_j}+g_j\overline{T'u'_i}), \tag{89}$$

where β is the coefficient of thermal expansion, and g_i is the acceleration due to gravity. The temperature-velocity correlation $\overline{T'u'_i}$ (also called the Reynolds heat flux) is modeled by a gradient transport hypothesis or is obtained from a modeled version of its transport equation.

While second-order closure models constitute, by far, the most promising approach in Reynolds-stress modeling, it must be said that they have not progressed to the point where reliable quantitative predictions can be made for a variety of turbulent flows. To illustrate this point, we again cite the case of rotating shear flow. As shown earlier, the phase-space portrait of second-order closures is far superior to that of any two-equation model of the eddy-viscosity type (i.e. the second-order closures properly predict that there is unstable flow only for an intermediate band of rotation rates; see Figure 8). However, the specific quantitative predictions of a wide variety of existing second-order closures were recently shown by Speziale et al (1989) to be highly contradictory in rotating shear flow for a significant range of Ω/S (see Figure 12). Comparable problems with the reliability of predictions when second-order closure models are integrated directly to a solid boundary persist, so that a variety of modifications—which usually involve the introduction of empirical wall damping that depends on the turbulence Reynolds number as well as the unit normal to the wall—continue to be proposed along alternative lines (cf Launder & Shima 1989, Mansour & Shih 1989, Lai & So 1990, Shih & Mansour 1990).

In the opinion of the present author, there are two major areas of development that are badly needed in order to improve the predictive capabilities of second-order closures:

1. The introduction of improved transport models for the turbulence length scale that incorporate at least some limited *two-point and directional information* (e.g. through some appropriate integral of the two-point velocity correlation tensor R_{ij}). In conjunction with this research, the use of gradient transport models should be reexamined. Although Donaldson & Sandri (1981) developed a tensor length scale along these lines, it was recently shown by Speziale (1990) that the specific form of the model that they chose can be collapsed to the standard ε-transport model in homogeneous flows.

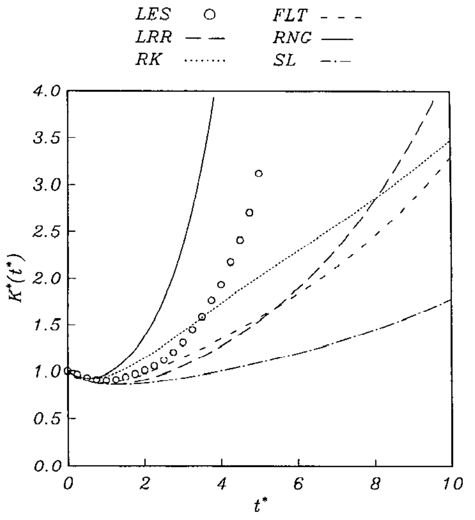

Figure 12 Comparison of the predictions of a variety of second-order closure models for the time evolution of the turbulent kinetic energy in rotating shear flow; $\Omega/S = 0.25$, $\varepsilon_0/SK_0 = 0.296$. LES ≡ large-eddy simulations of Bardina et al (1983); LRR ≡ Launder, Reece & Rodi model; RK ≡ Rotta-Kolmogorov model of Meller & Herring (1973); FLT ≡ Fu, Launder & Tselepidakis (1987) model; RNG ≡ Renormalization Group model of Yakhot & Orszag (1986); SL ≡ Shih & Lumley (1985) model.

2. The need for asymptotically consistent low-turbulence-Reynolds-number extensions of existing models that can be robustly integrated to a solid boundary. Existing models use ad hoc damping functions based on Re_t and have an implicit dependence on the unit normal to the wall that does not allow for the proper treatment of geometrical discontinuities such as those that occur in the square duct or back-step problems. Furthermore, the *nonlinear* effect of both rotational and irrotational strains need to be accounted for in the modeling of near-wall anisotropies in the dissipation.

In addition, the neglect of nonlocal and history effects in the commonly adopted models (74)–(75) for the pressure-strain correlation needs to be seriously reexamined. It has long been known that nonlocal effects can be quite important in strongly inhomogeneous turbulent flows. Furthermore, some inconsistencies that these algebraic models give rise to in rotating homogeneous turbulent flows have recently surfaced that appear to be due to the neglect of history effects in the rapid pressure-strain terms (see Reynolds 1989, Speziale et al 1990).

CONCLUDING REMARKS

There has been a tendency to be overly pessimistic about the progress made in Reynolds-stress modeling during the past few decades. It must be remembered that the first complete Reynolds-stress models—cast in tensor form and supplemented only with initial and boundary conditions—were developed less than 20 years ago. Progress was at first stymied by the lack of adequate computational power to properly explore full Reynolds-stress closures in nontrivial turbulent flows—a deficiency that was not overcome until the late 1960s. Then, by 1980—with an enormous increase in computer capacity—efforts were shifted toward direct and large-eddy simulations of the Navier-Stokes equations. Furthermore, the interest in coherent structures (cf Hussain 1983) and in alternative theoretical approaches based on nonlinear dynamics (e.g. period-doubling bifurcations as a route to chaos; cf Swinney & Gollub 1981) that crystallized during the late 1970s has also diverted attention away from Reynolds-stress modeling, as well as away from the general statistical approach for that matter. While progress has been slow, this is due in large measure to the intrinsic complexity of the problem. The fact that real progress has been made, however, cannot be denied. Many of the turbulent flows considered in the last section—which were solved without the ad hoc adjustment of any constants—could not be properly analyzed by the Reynolds-stress models that were available before 1970.

Some discussion is warranted concerning the goals and limitations of Reynolds-stress modeling. Under the best of circumstances, Reynolds-stress models can only provide accurate information about first and second one-point moments (e.g. the mean velocity, mean pressure, and turbulence intensity), which usually is all that is needed for design purposes. Since Reynolds-stress modeling constitutes a low-order one-point closure, it intrinsically cannot provide detailed information about flow structures. Furthermore, since spectral information needs to be indirectly built into Reynolds-stress models, a given model cannot be expected to perform well in a variety of turbulent flows where the spectrum of the energy-containing eddies is changing dramatically. However, to criticize Reynolds-stress models purely on the grounds that they are not based directly on solutions of the full Navier-Stokes equations would be as simplistic as criticizing exact solutions of the Navier-Stokes equations for not being rigorously derived from the Boltzmann equation or, for that matter, from quantum mechanics. The more appropriate question is whether or not a Reynolds-stress model can be developed that will provide adequate engineering answers for the mean velocity, mean pressure, and turbulence intensities in a significant range of turbulent flows that are of technological interest.

To obtain accurate predictions for these quantities in *all* possible turbulent flows will probably require nothing short of solving the full Navier-Stokes equations. Such a task will not be achievable in the foreseeable future, if ever at all (cf Hussaini et al 1990, Reynolds 1990). To gain an appreciation for the magnitude of this task, consider the fact that economically feasible direct simulations of turbulent pipe flow at a Reynolds number of 500,000—a turbulent flow that, although nontrivial, is far from the most difficult encountered—would require a computer that is 10 million times faster than the Cray YMP!

While second-order closures represent the most promising approach in Reynolds-stress modeling, much work remains to be done. The two problem areas mentioned in the previous section—namely, the development of transport models for an anisotropic integral length scale and the development of more asymptotically consistent methods for the integration of second-order closures to a solid boundary—are of utmost importance. In fact, the issue of near-wall modeling is so crucial that deficiencies in it—along with associated numerical stiffness problems—are primarily responsible for the somewhat misleading critical evaluations of second-order closures that arose out of the 1980–81 AFOSR-HTTM-Stanford Conference on Complex Turbulent Flows (see Kline et al 1981). Another area that urgently needs attention is the second-order closure modeling of compressible turbulent flows. Until recently, most compressible second-order closure modeling has consisted of Favre-averaged, variable-density extensions of the incompressible models (cf Cebeci & Smith 1974). However, with the current thrust in compressible second-order modeling at NASA Langley and NASA Ames, some new compressible modeling ideas—such as dilatational dissipation—have come to the forefront (see Sarkar et al 1989, Zeman 1990). Much more work in this area is needed, however.

Reynolds-stress modeling should continue to steadily progress, complementing numerical simulations of the Navier-Stokes equations and alternative theoretical approaches. In fact, with anticipated improvements in computer capacity, direct numerical simulations should begin to play a pivotal role in the screening and calibration of turbulence models. The recent work on Reynolds-stress budgets at NASA Ames constitutes an excellent example of this (see Mansour et al 1988). Furthermore, from the theoretical side, statistical mechanics approaches such as RNG could be of considerable future use in the formulation of new models. (Unfortunately, at their current stage of development, it does not appear that they can reliably calibrate turbulence models for use in complex flows.) Although Reynolds-stress models provide information only about a limited facet of turbulence, this information can have such important scientific and engineering applications that these models are likely to remain a part of turbulence research for many years to come.

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Literature Cited

Avva, R. K., Kline, S. J., Ferziger, J. H. 1988. Computation of the turbulent flow over a backward-facing step using the zonal modeling approach. *Rep. No. TF-33*, Stanford Univ., Stanford, Calif.

Baldwin, B. S., Lomax, H. 1978. Thin layer approximation and algebraic model for separated turbulent flows. *AIAA Pap. No. 78-257*

Bardina, J., Ferziger, J. H., Reynolds, W. C. 1983. Improved turbulence models based on large-eddy simulation of homogeneous, incompressible turbulent flows. *Rep. No. TF-19*, Stanford Univ., Stanford, Calif.

Bardina, J., Ferziger, J. H., Rogallo, R. S. 1985. Effect of rotation on isotropic turbulence: computation and modeling. *J. Fluid Mech.* 154: 321–36

Bernard, P. S., Berger, B. S. 1982. A method for computing three-dimensional turbulent flows. *SIAM J. Appl. Math.* 42: 453–70

Bertoglio, J. P. 1982. Homogeneous turbulent field within a rotating frame. *AIAA J.* 20: 1175–81

Boussinesq, J. 1877. Théorie de l'écoulement tourbillant. *Mém. Présentés par divers Savants Acad. Sci. Inst. Fr.* 23: 46–50

Bradshaw, P., Ferriss, D. H., Atwell, N. P. 1967. Calculation of boundary layer development using the turbulent energy equation. *J. Fluid Mech.* 28: 593–616

Bradshaw, P., Cebeci, T., Whitelaw, J. H. 1981. *Engineering Calculation Methods for Turbulent Flow*. New York: Academic

Cebeci, T., Smith, A. M. O. 1968. A finite-difference solution of the incompressible turbulent boundary-layer equations by an eddy viscosity concept. *Proc. AROSR-IFP-Stanford Conf. Comput. of Turbul. Boundary Layers*, ed. S. J. Kline, M. V. Morkovin, G. Sovran, D. S. Cockrell, 1: 346–55. Stanford, Calif: Stanford Univ. Press

Cebeci, T., Smith, A. M. O. 1974. *Analysis of Turbulent Boundary Layers*. New York: Academic

Choi, K. S., Lumley, J. L. 1984. Return to isotropy of homogeneous turbulence revisited. In *Turbulence and Chaotic Phenomena in Fluids*, ed. T. Tatsumi, pp. 267–72. New York: North-Holland

Clark, T. L., Farley, R. D. 1984. Severe downslope windstorm calculations in two and three spatial dimensions using anelastic interactive grid nesting: a possible mechanism for gustiness. *J. Atmos. Sci.* 41: 329–50

Comte-Bellot, G., Corrsin, S. 1971. Simple Eulerian time correlation of full- and narrow-band velocity signals in grid-generated, "isotropic" turbulence. *J. Fluid Mech.* 48: 273–337

Craft, T., Fu, S., Launder, B. E., Tselepidakis, D. P. 1989. Developments in modeling the turbulent second-moment pressure correlations. *Rep. No. TFD/89/1*, Mech. Eng. Dep., Univ. Manchester Inst. Sci. Technol. Engl.

Crow, S. C. 1968. Viscoelastic properties of fine-grained incompressible turbulence. *J. Fluid Mech.* 33: 1–20

Daly, B. J., Harlow, F. H. 1970. Transport equations in turbulence. *Phys. Fluids* 13: 2634–49

Deardorff, J. W. 1973. The use of subgrid transport equations in a three-dimensional model of atmospheric turbulence. *ASME J. Fluids Eng.* 95: 429–38

Demuren, A., Rodi, W. 1984. Calculation of turbulence-driven secondary motion in non-circular ducts. *J. Fluid Mech.* 140: 189–222

Donaldson, C. duP. 1972. Calculation of turbulent shear flows for atmospheric and vortex motions. *AIAA J.* 10: 4–12

Donaldson, C. duP., Rosenbaum, H. 1968. Calculation of the turbulent shear flows through closure of the Reynolds equations by invariant modeling. *Rep. No. 127*, Aeronaut. Res. Assoc. Princeton, Princeton, N.J.

Donaldson, C. duP., Sandri, G. 1981. On the inclusion of information on eddy structure in second-order closure model of turbulent flow. *AGARD Rep. No. CP-308*

Fu, S., Launder, B. E., Tselepidakis, D. P. 1987. Accommodating the effects of high strain rates in modelling the pressure-strain correlation. *Rep. No. TFD/87/5*, Mech. Eng. Dep., Univ. Manchester Inst. Sci. Technol., Engl.

Gatski, T. B., Savill, A. M. 1989. An analysis of curvature effects for the control of wall-bounded shear flows. *AIAA Pap. No. 89-1014*

Gessner, F. B., Po, J. K. 1976. A Reynolds stress model for turbulent corner flow—Part II: comparisons between theory and experiment. *ASME J. Fluids Eng.* 98: 269–77

Gibson, M. M., Launder, B. E. 1978. Ground effects on pressure fluctuations in the atmospheric boundary layer. *J. Fluid Mech.* 86: 491–511

Gibson, M. M., Rodi, W. 1981. A Reynolds stress closure model of turbulence applied to the calculation of a highly curved mixing layer. *J. Fluid Mech.* 103: 161–82

Gibson, M. M., Younis, B. A. 1986. Calculation of swirling jets with a Reynolds stress closure. *Phys. Fluids* 29: 38–48

Hanjalić, K., Launder, B. E. 1972. A Reynolds stress model of turbulence and its application to thin shear flows. *J. Fluid Mech.* 52: 609–38

Hanjalić, K., Launder, B. E. 1976. Contribution towards a Reynolds stress closure for low Reynolds number turbulence. *J. Fluid Mech.* 74: 593–610

Hanjalić, K., Launder, B. E. 1980. Sensitizing the dissipation equation to irrotational strains. *ASME J. Fluids Eng.* 102: 34–40

Haworth, D. C., Pope, S. B. 1986. A generalized Langevin model for turbulent flows. *Phys. Fluids* 29: 387–405

Hinze, J. O. 1975. *Turbulence*. New York: McGraw-Hill. 2nd ed.

Hussain, A. K. M. F. 1983. Coherent structures: reality and myth. *Phys. Fluids* 26: 2816–50

Hussaini, M. Y., Speziale, C. G., Zang, T. A. 1990. Discussion of the potential and limitations of direct and large-eddy simulations. *Proc. Whither Turbul. Workshop, Ithaca, N.Y. Lect. Notes Phys.*, ed. J. L. Lumley, pp. 354–68. Berlin: Springer-Verlag

Johnson, D. A., King, L. S. 1984. A new turbulence closure model for boundary layer flows with strong adverse pressure gradients and separation. *AIAA Pap. No. 84-0175*

Johnston, J. P., Halleen, R. M., Lezius, D. K. 1972. Effects of a spanwise rotation on the structure of two-dimensional fully-developed turbulent channel flow. *J. Fluid Mech.* 56: 533–57

Kim, J. 1983. The effect of rotation on turbulence structure. *NASA TM-84379*

Kim, J., Kline, S. J., Johnston, J. P. 1980. Investigation of a reattaching turbulent shear layer: flow over a backward-facing step. *ASME J. Fluids Eng.* 102: 302–8

Kline, S. J., Cantwell, B. J., Lilley, G. M., eds. 1981. *Proceedings of the 1980–81 AFOSR-HTTM-Stanford Conference on Complex Turbulent Flows*. Stanford, Calif: Stanford Univ. Press

Kolmogorov, A. N. 1942. The equations of turbulent motion in an incompressible fluid. *Izv. Acad. Sci. USSR, Phys.* 6: 56–58

Kraichnan, R. H. 1964. Direct interaction approximation for shear and thermally driven turbulence. *Phys. Fluids* 7: 1048–62

Lai, Y. G., So, R. M. C. 1990. On near wall turbulent flow modeling. *J. Fluid Mech.* In press

Launder, B. E. 1990. Phenomenological modeling: present and future. *Proc. Whither Turbul. Workshop, Ithaca, N.Y. Lect Notes Phys.*, ed. J. L. Lumley, pp. 439–85. Berlin: Springer-Verlag

Launder, B. E., Reece, G. J., Rodi, W. 1975. Progress in the development of a Reynolds-stress turbulence closure. *J. Fluid Mech.* 68: 537–66

Launder, B. E., Shima, N. 1989. Second-moment closure for the near-wall sublayer: development and application. *AIAA J.* 27: 1319–23

Launder, B. E., Spalding, D. B. 1974. The numerical computation of turbulent flows. *Comput. Methods Appl. Mech. Eng.* 3: 269–89

Launder, B. E., Tselepidakis, D. P., Younis, B. A. 1987. A second-moment closure study of rotating channel flow. *J. Fluid Mech.* 183: 63–75

Launder, B. E., Ying, W. M. 1972. Secondary flows in ducts of square cross-section. *J. Fluid Mech.* 54: 289–95

Lesieur, M. 1987. *Turbulence in Fluids*. Boston: Martinus Nijhoff

Lumley, J. L. 1970. Toward a turbulent constitutive equation. *J. Fluid Mech.* 41: 413–34

Lumley, J. L. 1978. Computational modeling of turbulent flows. *Adv. Appl. Mech.* 18: 123–76

Lumley, J. L. 1983. Turbulence modeling. *ASME J. Appl. Mech.* 50: 1097–1103

Lumley, J. L., Khajeh-Nouri, B. 1974. Computational modeling of turbulent transport. *Adv. Geophys.* 18A: 169–92

Mansour, N. N., Kim, J., Moin, P. 1988. Reynolds stress and dissipation rate budgets in turbulent channel flow. *J. Fluid Mech.* 194: 15–44

Mansour, N. N., Shih, T. H. 1989. Advancements in turbulence modeling. In *Forum on Turbulent Flows—1989*, ed. W. W. Bower, M. J. Morris, FED-76: 129–41. New York: ASME

Mellor, G. L., Herring, H. J. 1973. A survey of mean turbulent field closure models. *AIAA J.* 11: 590–99

Mellor, G. L., Yamada, T. 1974. A hierarchy of turbulence closure models for planetary boundary layers. *J. Atmos. Sci.* 31: 1791–1806

Moin, P., Kim, J. 1982. Numerical investigation of turbulent channel flow. *J. Fluid Mech.* 118: 341–77

Nisizima, S., Yoshizawa, A. 1987. Turbulent channel and Couette flows using an anisotropic *K*–ε model. *AIAA J.* 25: 414–20

Norris, L. H., Reynolds, W. C. 1975. Turbulent channel flow with a moving wavy boundary. *Rep. No. FM-10*, Stanford Univ., Stanford, Calif.

Orszag, S. A. 1970. Analytical theories of turbulence. *J. Fluid Mech.* 41: 363–86

Patel, V. C., Rodi, W., Scheuerer, G. 1985. Turbulence models for near-wall and low Reynolds number flows: a review. *AIAA J.* 23: 1308–19

Piomelli, U., Ferziger, J. H., Moin, P. 1987. Models for large eddy simulations of turbulent channel flows including transpiration. *Rep. No. TF-32*, Stanford Univ., Stanford, Calif.

Pope, S. B. 1975. A more general effective viscosity hypothesis. *J. Fluid Mech.* 72: 331–40

Pope, S. B. 1978. An explanation of the turbulent round jet/plane jet anomaly. *AIAA J.* 16: 279–81

Pope, S. B. 1985. Pdf methods for turbulent reactive flows. *Prog. Energy Combust. Sci.* 11: 119–92

Prandtl, L. 1925. Über die ausgebildete Turbulenz. *ZAMM* 5: 136–39

Prandtl, L. 1945. Über ein neues Formelsystem für die ausgebildete Turbulenz. *Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl.* 1945: 6–19

Reynolds, O. 1895. On the dynamical theory of incompressible viscous fluids and the determination of the criterion. *Philos. Trans. R. Soc. London Ser. A* 186: 123–64

Reynolds, W. C. 1976. Computation of turbulent flows. *Annu. Rev. Fluid Mech.* 8: 183–208

Reynolds, W. C. 1987. Fundamentals of turbulence for turbulence modeling and simulation. In *Lecture Notes for Von Kármán Institute. AGARD Lect. Ser. No. 86*, pp. 1–66. New York: NATO

Reynolds, W. C. 1989. Effects of rotation on homogeneous turbulence. *Proc. Australas. Fluid Mech. Conf., 10th*, pp. 1–6. Melbourne: Univ. Melbourne

Reynolds, W. C. 1990. The potential and limitations of direct and large-eddy simulations. *Proc. Whither Turbul. Workshop, Ithaca, N.Y. Lect. Notes Phys.*, ed. J. L. Lumley, pp. 313–42. Berlin: Springer-Verlag

Rivlin, R. S. 1957. The relation between the flow of non-Newtonian fluids and turbulent Newtonian fluids. *Q. Appl. Math.* 15: 212–15

Rodi, W. 1976. A new algebraic relation for calculating the Reynolds stresses. *ZAMM* 56: T219–21

Rodi, W. 1980. *Turbulence Models and Their Application in Hydraulics.* Delft, Neth: Int. Assoc. Hydraul. Res.

Rogallo, R. S., Moin, P. 1984. Numerical simulation of turbulent flows. *Annu. Rev. Fluid Mech.* 16: 99–137

Rotta, J. C. 1951. Statistische Theorie nichthomogener Turbulenz. *Z. Phys.* 129: 547–72

Rotta, J. C. 1972. Recent attempts to develop a generally applicable calculation method for turbulent shear flow layers. *AGARD CP-93*

Rubinstein, R., Barton, J. M. 1990. Nonlinear Reynolds stress models and the renormalization group. *Phys. Fluids A* 2: 1472–76

Saffman, P. G. 1977. Results of a two-equation model for turbulent flows and development of a relaxation stress model for application to straining and rotating flows. *Proc. Proj. SQUID Workshop Turbul. in Intern. Flows*, ed. S. Murthy, pp. 191–231. New York: Hemisphere

Sarkar, S., Erlebacher, G., Hussaini, M. Y., Kreiss, H. O. 1989. The analysis and modeling of dilatational terms in compressible turbulence. *Rep. No. 89-79*, ICASE, NASA Langley Res. Cent., Hampton, Va.

Sarkar, S., Speziale, C. G. 1990. A simple nonlinear model for the return to isotropy in turbulence. *Phys. Fluids A* 2: 84–93

Schlichting, H. 1968. *Boundary Layer Theory.* New York: McGraw-Hill

Schumann, U. 1977. Realizability of Reynolds stress turbulence models. *Phys. Fluids* 20: 721–25

Shih, T.-H., Lumley, J. L. 1985. Modeling of pressure correlation terms in Reynolds stress and scalar flux equations. *Rep. No. FDA-85-3*, Cornell Univ., Ithaca, N.Y.

Shih, T.-H., Mansour, N. N. 1990. Second-order modeling of low Reynolds number turbulence near walls. Submitted for publication

Smagorinsky, J. 1963. General circulation experiments with the primitive equations. *Mon. Weather Rev.* 91: 99–165

Smith, G. F. 1971. On isotropic functions of symmetric tensors, skew-symmetric tensors and vectors. *Int. J. Eng. Sci.* 9: 899–916

Smolarkiewicz, P. K., Clark, T. L. 1985. Numerical simulation of a three-dimensional field of cumulus clouds. Part I. Model description, comparison with observations and sensitivity studies. *J. Atmos. Sci.* 42: 502–22

Speziale, C. G. 1981. Some interesting properties of two-dimensional turbulence. *Phys. Fluids* 24: 1425–27

Speziale, C. G. 1983. Closure models for rotating two-dimensional turbulence. *Geophys. Astrophys. Fluid Dyn.* 23: 69–84

Speziale, C. G. 1985. Modeling the pressure gradient-velocity correlation of turbulence. *Phys. Fluids* 28: 69–71

Speziale, C. G. 1987a. Second-order closure models for rotating turbulent flows. *Q. Appl. Math.* 45: 721–33

Speziale, C. G. 1987b. On nonlinear *K-l* and *K-ε* models of turbulence. *J. Fluid Mech.* 178: 459–75

Speziale, C. G. 1989. Turbulence modeling in non-inertial frames of reference. *Theor. Comput. Fluid Dyn.* 1: 3–19

Speziale, C. G. 1990. Discussion of turbulence modeling: present and future. *Proc. Whither Turbul. Workshop, Ithaca, N.Y. Lect. Notes Phys.*, ed. J. L. Lumley, pp. 490–512. Berlin: Springer-Verlag

Speziale, C. G., Gatski, T. B., Mac Giolla Mhuiris, N. 1989. A critical comparison of turbulence models for homogeneous shear flows in a rotating frame. *Proc. Symp. Turbul. Shear Flows, 7th*, 2: 27.3.1–27.3.6. Stanford, Calif: Stanford University Press

Speziale, C. G., Mac Giolla Mhuiris, N. 1989a. On the prediction of equilibrium states in homogeneous turbulence. *J. Fluid Mech.* 209: 591–615

Speziale, C. G., Mac Giolla Mhuiris, N. 1989b. Scaling laws for homogeneous turbulent shear flows in a rotating frame. *Phys. Fluids A* 1: 294–301

Speziale, C. G., Mansour, N. N., Rogallo, R. S. 1987. The decay of isotropic turbulence in a rapidly rotating frame. *Proc. Summer Program Cent. for Turbul. Res.*, ed. P. Moin, W. C. Reynolds, J. Kim, pp. 205–11. Stanford, Calif: Stanford Univ. Press

Speziale, C. G., Ngo, T. 1988. Numerical solution of turbulent flow past a backward-facing step using a nonlinear *K-ε* model. *Int. J. Eng. Sci.* 26: 1099–1112

Speziale, C. G., Sarkar, S., Gatski, T. B. 1990. Modeling the pressure-strain correlation of turbulence—an invariant dynamical systems approach. *Rep. No. 90-5*, ICASE, NASA Langley Res. Cent., Hampton, Va.

Swinney, H. L., Gollub, J. P., eds. 1981. *Hydrodynamic Instabilities and the Transition to Turbulence.* New York: Springer-Verlag

Tavoularis, S., Corrsin, S. 1981. Experiments in nearly homogeneous turbulent shear flow with a uniform mean temperature gradient. Part I. *J. Fluid Mech.* 104: 311–47

Taylor, G. I. 1915. Eddy motion in the atmosphere. *Philos. Trans. R. Soc. London Ser. A* 215: 1–26

Tennekes, H., Lumley, J. L. 1972. *A First Course in Turbulence.* Cambridge, Mass: MIT Press. 300 pp.

von Kármán, T. 1930. Mechanische ähnlichkeit und Turbulenz. *Proc. Int. Congr. Appl. Mech., 3rd, Stockholm*, Pt. 1, pp. 85–105

von Kármán, T. 1948. Progress in the statistical theory of turbulence. *Proc. Natl. Acad. Sci. USA* 34: 530–39

Weinstock, J. 1981. Theory of the pressure-strain rate correlation for Reynolds stress turbulence closures. Part 1. *J. Fluid Mech.* 105: 369–96

Wigeland, R. A., Nagib, H. M. 1978. Grid-generated turbulence with and without rotation about the streamwise direction. *Rep. No. R78-1*, Ill. Inst. Technol., Chicago

Wilcox, D. C. 1988. Reassessment of the scale-determining equation for advanced turbulence models. *AIAA J.* 26: 1299 1310

Wilcox, D. C., Traci, R. M. 1976. A complete model of turbulence. *AIAA Pap. No. 76-351*

Wolfshtein, M. 1970. On the length scale of turbulence equation. *Isr. J. Technol.* 8: 87–99

Yakhot, V., Orszag, S. A. 1986. Renormalization group analysis of turbulence. I. Basic theory. *J. Sci. Comput.* 1: 3–51

Yoshizawa, A. 1984. Statistical analysis of the deviation of the Reynolds stress from its eddy viscosity representation. *Phys. Fluids* 27: 1377–87

Yoshizawa, A. 1987. Statistical modeling of a transport equation for the kinetic energy dissipation rate. *Phys. Fluids* 30: 628–31

Zeman, O. 1990. Dilatation dissipation: the concept and application in modeling compressible mixing layers. *Phys. Fluids A* 2: 178–88

Zeman, O., Lumley, J. L. 1976. Modeling buoyancy-driven mixed layers. *J. Atmos. Sci.* 33: 1974–88

Zeman, O., Lumley, J. L. 1979. Buoyancy effects in entraining turbulent boundary layers: a second-order closure study. In *Turbulent Shear Flows I*, ed. F. Durst, B. E. Launder, F. W. Schmidt, J. H. Whitelaw, pp. 295–302. Berlin: Springer-Verlag

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