Method of Manufactured Solutions

Biswajit Banerjee

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This is based on the following papers:

1) R. C. Batra and X. Q. Liang, 1997, ”Finite dynamic deformations of smart structures,” *Computational Mechanics*, **20**, 427–438.

2) R. C. Batra nd B. M. Love, 2006, ”Multiscale analysis of adiabatic shear bands in tungsten heavy alloy particulate composites,” *International Journal for Multiscale Computational Engineering*, **4**(1), 95–114.

1 Governing Equations

Let the motion be given by

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t)$$

where $\mathbf{x}(\mathbf{X})$ is the position of the material point $\mathbf{X}$ in the current configuration. The deformation gradient is given by

$$\boldsymbol{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \boldsymbol{\nabla}_o\, \mathbf{x} \ .$$

Let $\mathbf{u}(\mathbf{X})$ be the displacement of the material point given by

$$\mathbf{u}(\mathbf{X}, t) = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \ .$$

Then the deformation gradient can be written as

$$\boldsymbol{F} = \mathbf{1} + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} = \mathbf{1} + \boldsymbol{\nabla}_o\, \mathbf{u} \ .$$

The determinant of $\boldsymbol{F}$ is

$$J := \det(\boldsymbol{F}) \ .$$

The left Cauchy-Green deformation tensor is

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = (\mathbf{1} + \boldsymbol{\nabla}_o\, \mathbf{u})(\mathbf{1} + \boldsymbol{\nabla}_o\, \mathbf{u})^T = \mathbf{1} + \boldsymbol{\nabla}_o\, \mathbf{u} + (\boldsymbol{\nabla}_o\, \mathbf{u})^T + (\boldsymbol{\nabla}_o\, \mathbf{u})(\boldsymbol{\nabla}_o\, \mathbf{u})^T \ .$$

Let $\boldsymbol{\sigma}$ be the Cauchy stress and let $\boldsymbol{P}$ be the 1st Piola-Kirchhoff stress. Then,

$$\boldsymbol{\sigma} = J^{-1}\, \boldsymbol{P}\, \boldsymbol{F}^T \ .$$

Let the constitutive relation be given by

$$\boldsymbol{\sigma} = \frac{\mu}{J}(\boldsymbol{B} - \mathbf{1}) + \frac{\lambda}{J}\, \ln(J)\, \mathbf{1} \ .$$

The Lagrangian version of the momentum equation is

$$\boldsymbol{\nabla}_o \cdot \boldsymbol{P} + \rho_0 \mathbf{b} = \rho_0 \ddot{\mathbf{u}}$$

where ρ_0 is the density in the reference configuration, $\mathbf{b}$ is the body force (with appropriate units), and $\ddot{\mathbf{u}}$ is the material time derivative of the displacement.

The method of fictitious body forces involves assuming a displacement field over the body; finding the body force, initial conditions, and boundary conditions that fit that solution; and then using that body force and the computed BCs and ICs in the numerical algorithm to arrive at a solution. If the solution matches the assumed displacement, then we're good to go.

2 A 1-D example

Let's write the equations in 1-D. The motion is given by

$$x_1 = \varphi(X_1, t) \; .$$

The deformation gradient is

$$F_{11} = \frac{\partial x_1}{\partial X_1} = 1 + \frac{\partial u_1}{\partial X_1} \; .$$

The determinant of the deformation gradient is

$$J = \det(\boldsymbol{F}) = F_{11} = 1 + \frac{\partial u_1}{\partial X_1} \; .$$

The left Cauchy-Green deformation tensor is

$$B_{11} = 1 + 2\frac{\partial u_1}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_1}\right)^2 \; .$$

The Cauchy stress and the 1st Piola-Kirchhoff stress are related by

$$\sigma_{11} = \frac{1}{F_{11}} \, P_{11} \, F_{11} = P_{11} \; .$$

Let us assume that the constitutive model can be simplified to

$$\sigma_{11} = \frac{1}{2}\, C(B_{11} - 1) = \frac{1}{2}\, C \left[2\frac{\partial u_1}{\partial X_1} + \left(\frac{\partial u_1}{\partial X_1}\right)^2\right]$$

where C is an elastic constant. The momentum equation is

$$\frac{\partial P_{11}}{\partial X_1} + \rho_0 \, b_1 = \rho_0 \, \ddot{u}_1 \; .$$

Let us now assume that the displacement field is given by

$$u_1(X_1, t) = X_1^2 \, \sin(\omega \, t) \; .$$

Therefore,

$$\begin{aligned} \frac{\partial u_1}{\partial X_1} &= 2 \, X_1 \, \sin(\omega \, t) \\ \frac{\partial u_1}{\partial t} &= X_1^2 \, \omega \, \cos(\omega \, t) \\ \frac{\partial^2 u_1}{\partial t^2} &= -X_1^2 \, \omega^2 \, \cos(\omega \, t) \end{aligned}$$

Therefore, the deformation gradient is

$$F_{11} = 1 + 2 \, X_1 \, \sin(\omega \, t) \; .$$

The left Cauchy-Green deformation is

$$B_{11} = 1 + 4\,X_1\,\sin(\omega\,t) + 4\,X_1^2\,\sin^2(\omega\,t)\,.$$

The first Piola-Kirchhof stress is

$$P_{11} = \frac{1}{2}\,C\,[4\,X_1\,\sin(\omega\,t) + 4\,X_1^2\,\sin^2(\omega\,t)]\,.$$

Therefore,

$$\frac{\partial P_{11}}{\partial X_1} = 2\,C\,\sin(\omega\,t)[1 + 2\,X_1\,\sin(\omega\,t)]\,.$$

Plugging $\frac{\partial P_{11}}{\partial X_1}$ and $\frac{\partial^2 u_1}{\partial t^2}$ into the momentum equation, we get

$$2\,C\,\sin(\omega\,t)[1 + 2\,X_1\,\sin(\omega\,t)] + \rho_o\,b_1 = -\rho_0\,X_1^2\,\omega^2\,\cos(\omega\,t)\,.$$

Therefore, the body force is

$$\boxed{b_1 = -X_1^2\,\omega^2\,\cos(\omega\,t) - \frac{2\,C}{\rho_0}\,\sin(\omega\,t)[1 + 2\,X_1\,\sin(\omega\,t)]\,.}$$

If the bar is of length L, the boundary conditions are

$$\boxed{u_1(0,t) = 0 \qquad \text{and} \qquad u_1(L,t) = L^2\,\sin(\omega\,t)\,.}$$

The initial conditions are

$$\boxed{u_1(X_1,0) = X_1^2\,\sin(0) = 0\,.}$$

When you apply the body force $b_1(X_1,t)$ and the initial and boundary conditions, you should get a solution that matches the chosen function.

Note that all these have been done in a Lagrangian configuration. You could alternatively do the same assuming a function $\mathbf{u}(\mathbf{x},t)$ and transforming the equations accordingly.